PROOF IN ALGEBRA AT THE UNIVERSITY LEVEL: ANALYSIS OF STUDENTS DIFFICULTIES

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The reported research aims to investigate the difficulties encountered by university mathematics-major students when dealing with proof in Algebra. An exercise dealing with an equivalence relation and a subgroup, with an a-priori difficulty level compatible with the students' preparation in Algebra, assessed the students' abilities to use their definitions in some new situations and to construct simple proof steps based on mastery of concepts and their symbolic representations. According to performed analyses, students' difficulties depend on conceptual, logical and meta-mathematical factors. Some pedagogical implications are derived.

I- INTRODUCTION

In the Algerian mathematical curricula, proofs become a regular activity only at the university level. After the last university reform, the time for courses and exercises sessions has been shortened for the same subjects in comparison with before. This change resulted in difficulties for teachers on how to teach the same material in a shorter time. In mathematical courses for engineer students, many teachers eliminated proofs from both courses and exercises to devote the 'short' time left only to definitions and statements of main theorems and results. For mathematics-major students, this was considered to be not convenient by many teachers, so they kept dealing with proofs during the exercise activities.

The teaching of logic subjects has been for years a subject of disagreement among teachers, educators and researchers (for a survey, see Durand-Guerrier &al, 2012); it comes back to discussion in Algeria these past few years, after eliminating logic and even the conscious treatment of mathematical symbols from high school programs. At present, at the university, elements of logic are taught only by some teachers. The question comes over and over again, how we can teach elements of logic so that they can help understanding formal statements from a syntactic point of view but also, and mainly and basically, from a semantic point of view. Anyway, even if this is not taught in logic lessons, it must be dealt with somewhere. The activity of proving seems to me a good mean for this. When students deal with proof, they must use definitions and results, construct formal steps with arguments and link all this in the final product of the proof. Weaknesses and pitfalls in doing so may be an effect of how mathematical concepts and mathematics proof are taught.

This study is intended to investigate mathematics-major students' difficulties in proving in Algebra and what are they related to in the teaching of mathematics, logic and proof. In order to perform this investigation a theoretical toolkit was derived from different sources and an exam exercise (at the end of the second Algebra course) was arranged and used, whose content was an equivalence relation and a subgroup.

II- BACKGROUND AND THEORETICAL TOOLKIT

Mathematics education literature offers a wide spectrum of positions concerning: the relationships between the proving process, including the exploration phases, and proof as a product (as examples, see Hanna, 2000 and Pedemonte, 2007), and between argumentation and proof (with several, different positions: as examples, see Duval, 1991 and Boero, Douek, Ferrari, 2008); what is relevant in the teaching and learning of proof and proving, in particular the role of logic and logical skills in proving (see Durand Guerrier, 2008 and Tanguay, 2007) ; classifications of students behaviours (Harel & Sowder, 1998) and interpretations of difficulties met by them in proving (as examples, see Dreyfus, 1999, Weber, 2001 and Selden&Selden, 2003). Some contributions on the above issues, in particular on those concerning formal and semantic aspects of proof, come also from mathematicians reflecting on their activities (see Thurston, 1994).

According to the aim of my study, I have considered some theoretical tools and positions that can provide different, sometimes alternative lenses to deal with data collected in my investigation. I have chosen:

- Vergnaud's theory of conceptual fields (Vergnaud, 1991), to account for Algebra as a conceptual field and to deal with the problem of meaning in Algebra. We may recall that according to Vergnaud the mastery of concepts and conceptual fields depends on the mastery of reference situations, operational invariants and linguistic representations. In particular, Vergnaud points out the deep difference between: knowing definitions of concepts, on one side; and dealing with problem situations that need to use properties ("operational invariants") and representations of concepts, and to make reference to previously experienced "reference situations" (a crucial component of transfer), on the other.

- The importance of the conceptual (in Vergnaud's sense) mastery of (meta-mathematical) knowledge about what a theorem and a proof are, as a component of that "culture of theorems", which should be passed on to students in order to promote their awareness of the "rules of the game" in proving activities (see Boero, Douek, Morselli, Pedemonte, 2010).

- Duval's epistemological and cognitive distinction between argumentation and proof as formal derivation (Duval, 1991), and the (partly) opposite positions presented in some more recent papers (see for instance Boero, Douek & Ferrari, 2008), which make also reference to Thurston's position (Thurston, 1994) on the crucial role of semantic aspects in the proving process and in the checking and communication of proofs.

- The crucial role of logic and logical skills in proving (Durand-Guerrier, 2008;

Tanguay, 2007). In our educational perspective, we follow Durand-Guerrier & al. (2012, p. 370) concerning what we mean here by Logic:

> (...) logic as the discipline that deals with both the semantic and syntactic aspects of the organization of mathematical discourse with the aim of deducing results that follow necessarily from a set of premises. When we refer to logic as a subject, we mainly restrict ourselves to the mathematical uses of the words *and*, *or*, *not*, and *if-then* (the basis for "propositional logic"), especially in statements that involve variables, as well as *for-all*, and *there-exists* (the extension to "predicate logic")

The resulting theoretical toolkit is not homogeneous; this choice depends on the exploratory character of this study and on the need of testing different lenses to interpret students' behaviours and difficulties.

III-METHODOLOGY

A-priori Analysis

The experiment is about an exercise (presented below) that has been chosen by me as a teacher of twenty math-major students at the second year university level. It was an exercise of an exam of the first semester (2011). The course (Algebra II) is about algebraic structures, it is taught during two sessions per week, one course session and one exercise session; each session lasts one hour and a half. The students were supposed to have been provided with the principal prerequesites in the previous year, in the Algebra I course (set theory, applications: surjections, injections; group theory). However, according to them, many of those concepts were not well mastered because of the little time devoted for each chapter; moreover dealing with proof was not among their regular tasks within the exercises proposed to them in Algebra I. I wanted, by choosing to assess students' preparation during an exam, to get the best possible results from them (students usually do prepare well to the exams) and I believe that working individually might reveal some errors that cannot appear when working in groups. I also wanted to know where (content, skills, ways of thinking) the difficulties with proof were situated - the answer was not clear for me during my semester teaching, especially when dealing with proofs (most exercises were about proofs). And the answer is crucial if we want to improve our teaching in order to help students overcome proof difficulties. Another issue to be dealt with was about the obstacle inherent in the mastery of formalisms and the relation with the teaching of logic. In mathematics courses, students show more and more difficulties in dealing with symbols (as concerns both syntax and semantics). Many teachers argue in favour of the elimination of teaching logic (in its technical aspects) because it is useless; many others insist on it as an initiation to mathematical language in such a way that students may interiorize some rules that can help them in understanding concepts, definitions, theorems, that intervene in proof production. The toolkit arranged for this study and presented in the previous Section was expected to help tackling the above problems.

I have chosen questions with high formal-symbolic features to get the students writing formally (which is normal at the university level and even the objective of some chapters of the courses for mathematics-major students) but also to check what reasoning resulted in correct formal proofs and what kinds of difficulties were met by students. Productive reasoning, I assume, is strongly based on concept understanding and argumentation skills. My aim was to investigate the movement, made during the proving process, between the concepts, the formal transcription and the argumentation. Within such a wide exploratory study I have chosen to present in this paper only data and analyses concerning students' difficulties in proving.

> Let G be a non commutative multiplicative group.
> Let R be a relation defined in G by: $x\ R\ y \Leftrightarrow \exists\ g \in G$ such that $y=gx$.
> 1-Prove that R is an equivalence relation.
> Let $x \in G$, we define G_x by: $G_x=\{g$ such that $gx=x\}$, prove that :
> 2- G_x is a subgroup of G.
> 3-The application defined in G by $f_g(x)=gxg^{-1}$ transforms a subgroup into a subgroup.

According to what students were supposed to have learned in their past algebra courses and exercises, the proofs should have been as follows.

> 1- Let's prove that R is reflexive, symmetric and transitive.
> Let $x \in G$, let's prove that xRx. Let's find a g in G that verifies $gx=x$, for $g=1$, we have $x=1x$. Hence $\forall x \in G$, xRx. That is R is a reflexive relation.
> Let be $x,y \in G$ such that xRy, let's prove that yRx. We have xRy i.e $\exists\ g \in$ G such that $y=gx$, as G is a group, then $x=g^{-1}y$ i.e yRx. Hence R is a symmetric relation.
> Let be $x,y,z \in G$ such that xRy et yRz, let's prove that xRz. We have xRy and yRz i.e $\exists\ g \in G$, $y=gx$ and $\exists\ g' \in$ G, $z=g'y$, by replacing y, we get $z=g'gx=g''x$ and $g'' \in G$ i.e xRz. Hence R is a transitive relation.
>
> 2-Let's prove that G_x is a subgroup of G.
> Let's prove that $e \in G_x$, we have $x=ex$, i.e $e \in G_x$ which means that G_x is not empty.
> Let's prove that if $g \in G_x$ then its inverse $g^{-1} \in G_x$. Let $g \in G_x$ i.e $gx=x$, multiplying by g^{-1} we get $x=g^{-1}x$ that is $g^{-1} \in G_x$.
> Let's prove that G_x is close under multiplication of G. Let g_1, $g_2 \in G_x$, let's prove that $g_1\ g_2 \in G_x$.
> We have $g_1\ x=x$ and $g_2\ x=x$, by replacing x in the first equality (or the second), we get $g_1g_2x=x$ which means that $g_1\ g_2 \in G_x$.. Hence G_x is a subgroup of G.

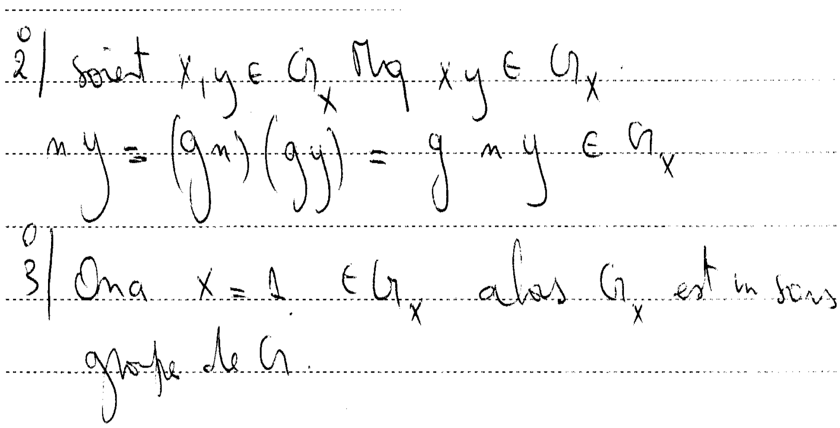
When a-priori evaluating the task, according to the exercises given to the students and both Algebra I and Algebra II courses, we arrived at the following conclusions (partly put into question by the analysis of students' performances - see Global Analysis):

- It is an easy exercise as the questions are classical and most exercises of equivalence relations and groups given to the students had the same questions.
- The mathematical content engaged in the proof is supposed to have been taught in the previous course (algebra I): groups, applications, direct image of a set by an application.
- The proofs of the statements are a direct application of definitions; there are no tricks or unusual techniques that may cause the students to be stuck.
- The students are expected to show that they are able: to understand and use the symbolic language (quantifiers, implication); to make the difference between a variable and a parameter; but especially to be able to use a definition in a new situation, which (according to Vergnaud) needs to master the concept beyond its definition and may allow to build the proof steps through its properties.

IV- ANALYSIS OF STUDENTS' DIFFICULTIES

I have chosen three individual productions, which will represent the situation of the great majority of students and the most common difficulties met by them.

Production 1


2°/ Soient $x, y \in G_x$ Mq $xy \in G_x$

$xy = (gx)(gy) = gxy \in G_x$

3°/ On a $x = 1 \in G_x$ alors G_x est un sous groupe de G.

This production deals with question 2 of the exercise.

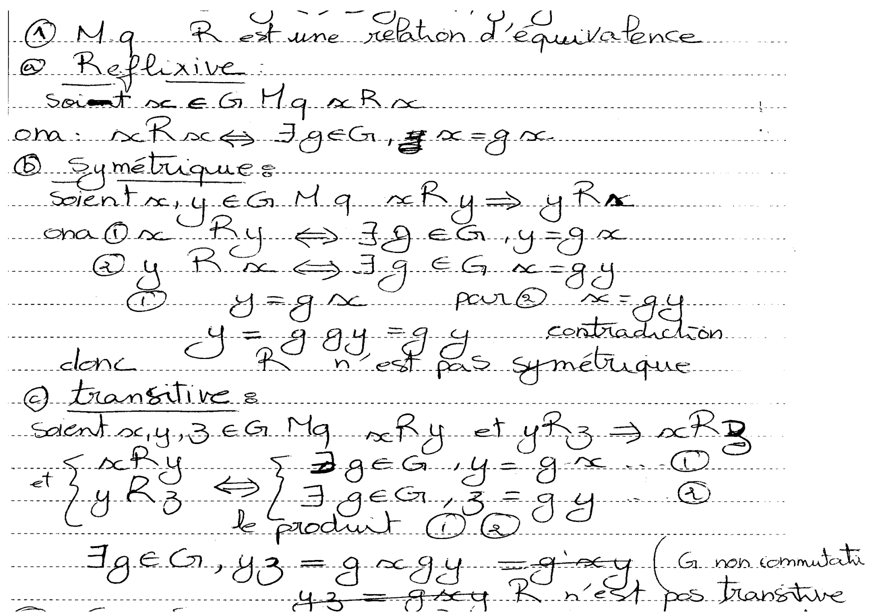
The student satisfies only two conditions among three for a subgroup, failing to prove that G_x contains the inverses of its elements. Despite of this, he finishes his proof by deducing the required result that is G_x is a subgroup; this might indicate how his conceptual (meta-) knowledge about proof is weak.

He starts by showing that G_x is closed for multiplication; he writes the definition as it is given in courses and mathematics books (with x and y) without adapting it to the exercise notations where the elements of G are denoted by g. Designing by x the elements of G is not convenient at all as x has already been used for the definition of G_x. The student shows a lack at the operatory level of transfer, probably related to an insufficient mastery of the concept of group (in the sense of Vergnaud).

In the second line, when trying to prove that the product of any two elements of G_x lies in it, the student replaces x by gx and y by gy; this shows that the elements of G_x are not clear for the student, and he is not able to identify the role of g. Here, the lack concerning the mastery of the conceptual content of the definition is clear. The last step (in line 2) is not justified by the student. I think that, being in an impasse, he wrote the last result just to finish, without being able to derive it from the previous steps. Again the lack of knowledge, of what a proof is, seems evident here.

In the third line, the lack at the operatory level of transfer is shown again by using *1* for x.

Production 2



In this production, we deal with question 1 of the exercise.
We can see how the student proves the opposite of the required result.

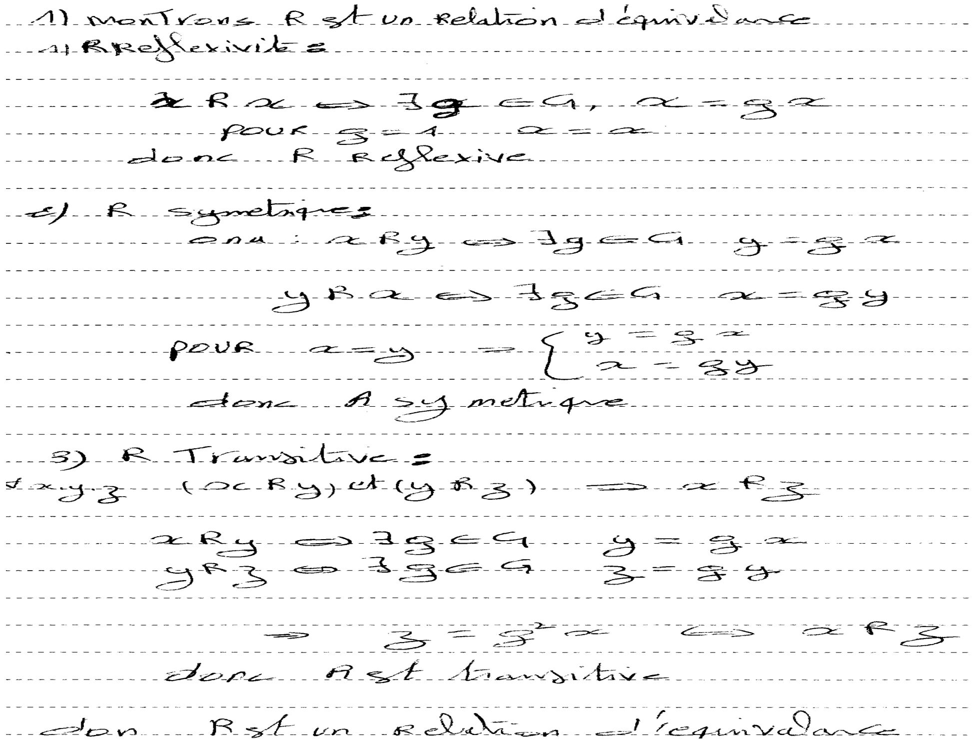
At the first step (reflexivity), the student writes correctly the definition of reflexivity and what is to be shown, but then his adding nothing, shows that he is not able to adapt it to the definition given in the exercise. Lack at the operatory level of transfer, which probably depends on insufficient mastery of the concept (in Vergnaud's sense), is clear here. At the second step (symmetry), the definitions and what is to be proved are also written correctly. The student starts proving by developing both the hypothesis (xRy) and the result (yRx) of the implication according to the given definition. Then what is supposed to be proved is used as a mean to make the proof. Even though, the student uses the same notation for g for both definitions, this doesn't show whether it is the same or not. When arriving to an impasse ($y=gy$), the student declares that it is a contradiction and deduces the opposite result. It seems to him the 'legal' way to get away. An important question arises here. Why he does not put into question his proof instead of questioning the required result? Why does he trust his reasoning more than the text of the exercise? I think that the student is not able to see other possibilities in his proof than what he could write; if this interpretation is correct, available data show how the student's logic abilities of understanding (semantic sense) and dealing with formal statements are very limited.

At the third step, the student begins as before, by showing that his definitions are very well memorised. Then the product of the two equations is given in order to prove xRz.

One cannot say for sure if g is considered to be the same or not. The existential quantifier, at the last line, indicates that g is the same, but in what is crossed out (g'), we can tell that it is not the same. So, the student is lost again (on the logical ground, in the sense of Durand-Guerrier & al, 2012, p. 670) and gets away as before.

We can see also that this student is far from having conceptual (meta-) knowledge about what a mathematical proof is from a semantic point of view.

Production 3



1) montrons R est un relation d'équivalance
1) R Reflexivité:
$xRx \Leftrightarrow \exists g \in G, x = gx$
pour $g = 1$ $x = x$
donc R Reflexive
2) R symetrique:
on a : $xRy \Leftrightarrow \exists g \in G$ $y = gx$
$yRx \Leftrightarrow \exists g \in G$ $x = gy$
pour $x = y \Rightarrow \begin{cases} y = gx \\ x = gy \end{cases}$
donc R symetrique
3) R Transitive:
$\forall x, y, z$ (xRy) et $(yRz) \Rightarrow xRz$
$xRy \Leftrightarrow \exists g \in G$ $y = gx$
$yRz \Leftrightarrow \exists g \in G$ $z = gy$
$\Rightarrow z = g^2 x \Leftrightarrow xRz$
donc R est transitive
don R est un relation d'équivalance

This production deals with the question 1 (equivalence relation).

The student tries to prove that the three conditions are satisfied. He starts good, at the reflexivity step, by replacing g by 1 to prove that xRx. But then he fails later, by considering the same g. This leads necessarily to an absurd situation (from $y=gx$ we want to get $x=gy$); to get away from this situation, a small and simple trick seems to be the key: putting $x=y$, without realising that this is in contradiction with the symmetry definition that is set for all elements. We can interpret this as due to a lack of logical-semantic mastery of the symbolic language.

At the third and last step, the student considers g to be the same as he gets ($z=g^2x$) when doing the product. But this is still a trap, because as g is the same, g^2 wouldn't work to deduce the last result, but despite of this, the student concludes by (xRz).

This behaviour on one side remembers the ritual proof scheme of Harel & Sowder (1998), on the other puts into question again the logical-semantic mastery of the symbolic language of the exercise.

Global analysis

The tasks proposed in the exercise that were supposed easy (as they belong to past courses activities) turned out to be complicated and some of them even impossible to deal with by students, given the level of their competencies revealed by the performed analysis (see below). From twenty students, no one could completely solve the first part of the exercise (equivalence relation), only two could give correct proof of some properties of the subgroup (question 2) and no one has dealt with the third question (the direct image of a subgroup). Three main problems have been shown in the productions: lack of transfer at the operatory level (depending on lack of mastery of the concept at stake, in Vergnaud's sense), lack of logical mastery of symbolically-stated definitions and inference steps (see Durand-Guerrier et al, 2012, p. 670), and lack of (meta-) knowledge about what a proof is in mathematics. The definitions are memorized only in a formal and very superficial way which is far to be sufficient when we need to adapt them to any other new situation, and even the logical mastery of their symbolic presentation is lacking. Moreover, available data confirm that students face strong difficulties in proofs involving multiple quantifiers (e. g. Chellougui, F., 2009). Still concerning proof, there is evidence that several students do not know what they are expected to do in a mathematical proof, how to use definitions, which they are provided with, and what the meaning of the hypothesis and the thesis is as elements related to the proving process. Most of them do nothing or meaningless operations and then deduce the final result. And some students seem not to know what "doing mathematics" means in general!

The undergraduate students tested are in their second university year and first year in mathematics as a discipline (math-majors). For these future teachers, it's clear that dealing with a proof seems to be new and strange. The formal-symbolic language is considered as a meaningless drawing for definitions. Making proofs shows well how it is difficult, even impossible, to deal with a concept by considering only the formal transcription without mastering the meaning. The semantic side of a proof seems also inexistent; it is thought that a proof should be a sequence of formal steps done by any technique, provided that it looks at the end, somehow, like the final result (cf. "ritual proof scheme" in Harel & Sowder, 1998). The students show difficulty with proof dealing with one single concept, even if they manage the beginning and some of the steps of the proof; but they are not able to start a proof that deals with many concepts (question 3: sets, direct image of a set by an application and subgroups).

V- CONCLUSION

The analysis shows a disconnection between formal transcription and conceptual mastering of the content of definitions. This makes me saying that at present the teaching of mathematics is more focused on formal-symbolic level rather than on the semantic level of mathematics content; on the contrary it should be necessary to teach definitions associated with all their conceptual content (in Vergnaud's sense: reference situations, operational invariants, linguistic representations) as well as their role and their usage in proving. The productions show students not being used to examine deeply every time the meaning of the definitions and the meaning of the formal symbols back and forth. This activity is crucial in proving and without it; the students cannot be able to learn how to make proofs. This might be taught partially in logic (in the sense of Durand-Guerrier & al., 2012, p. 370) and so can help in the future mathematics activity.

We have also seen that the absence of meta-knowledge about the proof in mathematics prevents to construct arguments that result in a proof, this may be a consequence of a teaching process that uses proofs as mean to set theorems and results and not as an object of an activity (as multiplication, continuity, integration or else) to be taught, used and mastered by students. Even when teachers make a proof at the board, most of them write formally the main steps and fail 'to write' the arguments justifying the inference steps, in order to show all that lies behind. I think as far as this is not revealed and shown clearly, the students cannot be aware of how the proving process works, which is mainly based on argumentation aimed at a deductive chain of propositions and inference steps related to their full meaning (cf. Thurston, 1994).

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