THE EPISTEMIC STATUS OF FORMALIZABLE PROOF AND FORMALIZABILITY AS A META-DISCURSIVE RULE

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The first two parts of this article report on a study that was part of my dissertation project at the interface of epistemology and sociology of mathematics. The study deals with the epistemic role of formalizability, which is traditionally held to be the main epistemic feature of mathematical proofs, in actual mathematical (research) practice. As a core result, it is argued that formalizability should be understood as a feature of discursive proving actions as the true bearers of epistemic value. As I discuss in the last part of the article, this insight opens the way for a shift to an educational perspective on proof in mathematical research practice. Sfard's approach to mathematical thinking as communication, with the concept of meta-discursive rules in particular, serves well as a conceptual framework to that end.

Keywords: mathematical proof, formalizability, epistemology, sociology, mathematical practice, meta-discursive rules

INTRODUCTION

In science education, the so-called nature of science is a widely agreed upon aspect of scientific literacy that one has to teach more or less explicitly. Important components of nature of science are epistemic features of scientific inquiry, the epistemic status of laws and theories, the tentativeness of science, the theory ladenness of observation, etc. (Akerson, Abd-El-Khalick, & Lederman, 2000). Regarding mathematical research practice and university education, proof appears to be an essential component of a "nature of mathematics". But are the essential epistemic features of proof in mathematical research practice relevant to school mathematics? If so, how could we teach them? Though these questions can only be touched in the last part of this article, they indicate an overall framework for employing the results from research on proof in actual mathematical research practice presented here, under appropriate re-interpretation where required, in thinking about teaching proof in school mathematics.

The work (Müller-Hill, 2011) that is presented in the first two parts of this paper was concerned with the epistemic role of formalizability of mathematical proofs in actual mathematical practice, with a major focus on research mathematics and a minor on mathematics education at university[1]. Regarding these contexts, one is traditionally inclined to demand formalizability—usually without further specification—as an essential epistemic feature of proof. Hence the main research question was:

> In what sense of "formalizable" is formalizability an essential epistemic feature of proof in actual mathematical practice, and thus a necessary condition for accepting a proof?

Formalizability is a feature of informal mathematical proofs: A formalizable proof is a proof that can be transferred into a formal proof, that means it can be transferred into a formal derivation with respect to a formal axiomatic system with consistent axioms. However, this notion of formalizability is not sufficiently specified. Possible semantics of the phrase “formalizable proof” still vary in a spectrum spread between two extremes. One extreme would be the weak reading “a proof of p is formalizable iff the informally proven mathematical theorem is also formally derivable in a consistent formal axiomatic system”; the other extreme is the strong reading “a proof of p is formalizable iff it can be translated step by step into a formal proof”, which may refer to, for example, proofs that are written in some semi-formal language.

Formalizability is indeed an important feature of mathematical proofs, regarding foundational issues in the philosophy of mathematics[2]. Foundational issues, however, are not addressed in the following. What is addressed instead is the epistemic role of this feature, from the viewpoint of a socio-empirically informed philosophy of mathematics.

SOCIO-EMPIRICALLY INFORMED PHILOSOPHY OF MATHEMATICS

The epistemic role of formalizability has traditionally been investigated by analytical philosophy of mathematics. The focus of such investigations is on the project of conceptually grasping formalizability as an epistemic feature of proof. The default method of analytical philosophy is the semantic analysis of so-called ordinary language epistemic concepts like knowledge and justification. It investigates the adequateness of having a formalizable proof as a truth-condition for, e.g., knowledge attributions. The analytical philosopher thereby almost exclusively relies on his professional intuition as an expert for epistemic concepts like knowledge, belief, or justification (see recently, e.g., Glock, 2008, for an introduction to analytical philosophy).

However, concerning the question of the epistemic role of formalizable proof in actual mathematical practice, an investigation of decidable conditions under which a proof is actually acceptable because of being formalizable appears to be equally worthwhile (see also Moser, 1991).

A socio-empirically informed philosophy of mathematics aims at (ideally) establishing a reflective equilibrium between the outcomes of both kinds of investigation. An appropriate methodological framework to this end is conceptual modelling as developed in (Löwe & Müller, 2011) and (Löwe, Müller, & Müller-Hill, 2010). Conceptual modelling “of X” takes the form of an iterative process:

> *Step 1 Theory formation* Guided by either a pre-theoretic understanding of X or the earlier steps in the iteration, one develops a structural philosophical account of X, including, e.g., considerations of ontology and epistemology.

Step 2 Phenomenology With a view towards Step 3, one collects data about X and extracts stable phenomena from them to corroborate or to question the current theory.

Step 3 Reflection In a circle between the philosophical theory, the philosophical theory formation process and the phenomenology, one assesses the adequacy of the theory and potentially revises the theory by reverting to Step 1.

In particular contrast to analytic epistemology, a socio-empirically informed epistemology of mathematical practice strengthens step 2 by including data established via accepted empirical methods from empirical sociology.

DESIGN OF AN INTERVIEW STUDY AS PART OF STEP 2

The conceptual modelling cycle displayed above was employed iteratively in my study (Müller-Hill, 2011). As one part of step 2, I developed and conducted a qualitative, so-called problem-centered, semi-standardized guideline interview study (see, e.g., Mayring, 2002) among research mathematicians. The aim was to gather detailed empirical information about practitioner's interpretations of the epistemic role of formalizable mathematical proof in actual mathematical practice.

The interview guideline (see Müller-Hill, 2011, 148 ff.) was developed with respect to certain key aspects that came out of earlier iterations of the modelling cycle, including in particular a quantitative questionnaire study reported in (Müller-Hill, 2009, 2011) on the use of epistemic attributions in actual mathematical practice.

Six mathematicians of high standing, from various fields of professional specialisation areas in mathematical research practice, from different countries and different institutions, were chosen as interviewees. The interviews were audio-recorded and transcribed. As the method of data analysis, I chose so-called phenomenological analysis. The method of phenomenological analysis, in a nutshell (cf. Mayring, 2002), is to reconstruct units of meaning in a sufficient variety of subjective interpretations and viewpoints of the matter in question as a first step. In a second step, these subjective, idiosyncratic units of meaning are synthesized, and reduced to a common, invariant essential core.

RESULTS AND INTERPRETATION

In the following, I will present some exemplary quotes from the interview transcripts, a short summary of the philosophical interpretation based on the analysis of the transcripts, and a central aspect of the subsequent conceptual theory formation (see Müller-Hill (2011) for more details on the data and its analysis).

Example quotes from the transcripts

The following examples stem from four of the six interviews.

Excerpts from interview 2:

Interviewer: And what would you call a formalizable proof?

IP 2: […] If something is obviously non formalizable, then to me it will be obviously not an acceptable proof.

Interviewer: Would you say that every proof that is accepted by the community is formalizable?[3]

IP 2: […] That's my belief, and one should say this is a hygienic belief – I mean, that's more or less a definition of what we, as mathematicians, as a community, are thinking about […] when we are talking about proofs.

Excerpts from interview 3:

Interviewer: Would you say that the proof about the classification of finite simple groups is a formalizable proof?

IP 3: […] I think in principle, it's formalizable, this classification.

Interviewer: What does 'in principle' mean?

IP 3: With the present technology, or just by hand, it may not be doable for one person in a lifetime and maybe even for the group of people who have been working on this it may not be doable, but with advanced future computer technology it may be doable.

Excerpts from interview 4:

IP 4: So one thing: maths is in a sense something personal. […] When do I think I understand something? Certainly if it's a relevant thing, during the next few weeks I would try to shoot holes in it, look at it from different directions, through different angles, ask questions why it is true, why it works this way, not that way. […]

Interviewer: What would be formal proof for you?

IP 4: […] I have to be able to understand both the global picture, to have an overview of the proof, explain to myself what the global idea is and why it works like this, and also to be able, and that's the details, to follow the proof from step to step, the logical consequences.

Excerpts from interview 5:

Interviewer: What do you mean by 'formal proof'?

IP 5: […] One of my recent graduate students is almost incapable of writing down proofs. But he really knows mathematical truths. The problem is then to extract from him why it is true, which is the proof. […] There are well respected mathematicians who are like this.

Interviewer: Can you make any sense of the term 'formalizable proof'?

IP5: It very much depends on the style of a mathematician, on a personal temper maybe, the attitudes of particular mathematicians, but it depends also on the field of mathematics. […]

Interviewer: Would you say that every proof that is accepted by the community is formalizable?

IP 5: […] Maybe I shouldn't think of myself as a mathematician if I don't believe that the proofs are formalizable. I think we act on assumption that the proofs that we produce are formalizable.

Excerpts from interview 6:

IP 6: When doing my Ph.D. thesis, I learned [...] something about the way to be careful […] If you see a proof, then you start reading it, and say, o.k., can I understand all the implications. This can be very clear, or can sometimes be a little bit blurred in that you are at the point of sort of believing it rather than actually seeing it. […]

Interviewer: Would you say that every proof that is accepted by the community is formalizable?

IP 6: […] I think they look like formalizable proofs. I don't know how they do that, but I think that all the people who have looked at these proofs think they are formalizable.

Summary of the philosophical interpretation

The philosophical interpretation developed in (Müller-Hill, 2011) on the basis of the phenomenological analysis of all interview transcripts, including those that are not displayed above, can be summarized and condensed into eight main aspects. I will only refer to the following six of these aspects here (see Müller-Hill, 2011, 205ff., for the whole list), with an emphasis on aspect (5) as a major conceptual turning point in the philosophical understanding of proofs as bearers of epistemic value.

1. In actual mathematical practice, acceptable proof includes the inevitable possibility of error. Hence the whole proving process has no final point, and epistemic attitudes of practitioners are not categorical. Nevertheless, there exist well working, situative standards for the error robustness of the core argumentation of an acceptable proof.
2. Equally besides and even beyond the aim of secure knowledge-that, mathematical practice aims at understanding in the sense of knowledge-why.
3. Explanatory proofs contribute essentially to knowledge-why. Hence, depending on the professional skill level of the epistemic subjects, acceptable proofs ought to have a meaningful argument structure. To that end, they often rely on established meta-argumentations that are not formalizable mutatis mutandis, or by mechanical means.
4. Epistemic standards for acceptable proof are gradual and context-dependent. They concern surveyability, clarity of the core argumentation, error robustness and formal correctness, the use of meta-argumentations and the possibility of perspective change.
5. Formalizability of acceptable proof should consistently be interpreted rather as an essential epistemic feature of the embedding communicative action than of the static,

linguistic argument presented as a proof. (This is not to say that the mere linguistic argument presented as a proof of *p* does not bear *any* epistemic value.)

6. The concept of formal proof functions as an abstract, internalized model of proof shaping the self-image of practising mathematicians. Sophisticated mathematicians may have internalized the rules and principles of formal proving and work in agreement with these rules without explicitly and consciously employing them.

Formalizability as an epistemic feature of discursive proving actions

Moving from these results of step 2 towards step 3 of the modelling cycle, theory formation, the main question is how to specify the notion of formalizability as an epistemic feature of mathematical proof that properly fits the empirical findings. According to the interpretation of the results of the interview study, there are several essential aspects of acceptable proof in actual mathematical practice, with formalizability as one of them. However, some of these essential aspects appear to compete against certain others, such as formalizability and fallibility, formalizability and the frequent use of sophisticated meta-argumentations, or formalizability and explanatory power. Hence, to form a consistent notion of formalizability as an epistemic essential of proof within an epistemology of mathematics needs to supersede the concept of proof itself as a mere linguistic entity—an argument—by a proper alternative[4].

Such an alternative account of an epistemologically relevant notion of proof in mathematical practice that can consistently be seen as a bearer of the partly competing epistemic aspects mentioned above is provided, as I argue and develop in detail in (Müller-Hill, 2011), by the notion of discursive proving actions. I will give a brief sketch of this conception of formalizability in the following.

My account makes reference to the concept of informing dialogical communicative actions used in philosophy of language and linguistics (cf. Meggle, 1999):

> An *informing dialogical communicative action* is an intentional act of communication in dialogue, with the communicative aim to make the receiver believe a certain thing.

Formalizability can thus be conceptualized as an epistemic feature of mathematical proof, in the sense that an epistemic subject *X* is justified to believe in the validity of a theorem *p* on the basis of an accepted proof, if *X* is able to carry out—given appropriate conditions—a certain kind of discursive proving action. I call these discursive proving actions “derivation indicating”[5]:

> A *discursive proving action* for a mathematical theorem *p* is an informing dialogical communicative action (oral or written) where an epistemic subject *X* presents an argumentation for the validity of *p* to a certain audience under certain situative circumstances. The presentation of the argumentation includes contributing utterances of members of the audience. A discursive proving action is called *derivation indicating* iff the type of argument, the presentation of the argument, and the professional level of the audience meet the contextually given epistemic standards and *X* has sufficient

mathematical skills to produce a appropriately formalized argument out of the given presentation. This may involve modification, correction and supplementation of the given presentation up to certain, context-sensitive levels of error robustness and stability of the core argument. (Müller-Hill, 2011, 230 f., German in the original)

FORMALIZABILITY AS A META-DISCURSIVE RULE

The general conceptual turn from mere linguistic entities to discursive proving actions in analyzing the essential epistemic features of mathematical proofs opens the way for using certain perspectives and theoretical frameworks from research in mathematics education for a re-interpretation of the results of the presented study. If this is successful, it could be a fruitful interface between a better understanding of the nature of proof in actual mathematical practice and a didactically sustainable way of teaching the nature of mathematical proof in the classroom.

Regarding research in mathematics education, Sfard's approach of mathematical thinking as communication and her concept of meta-discursive rules (Sfard, 2001, 2002, 2007, 2008) seem to grasp the core of my socio-empirically informed account of the epistemic role of formalizability particularly well. According to Sfard, meta-discursive rules, in contrast to object-level discursive rules, are rules about the discourse[6]. Formalizability, understood as a feature of discursive proving actions, can be seen as such a meta-discursive rule in actual mathematical practice. Within the scope of this article, I can only highlight two characteristics of formalizability as a meta-discursive rule in the sense of Sfard.

First, according to Sfard (2002, 30) "meta-rules are usually not anything the interlocutors would be fully aware of, or would follow consciously". Rather, "in concert with meta-discursive rules, people undertake actions that count as appropriate in a given context and refrain from behaviours that would look out of place". This characterization of meta-discursive rules fits well with aspect (6), formal proof as an internalized leading picture, aspect (3) regarding the role of mathematical professional skills, and aspect (4), context-sensitive epistemic standards, from the general interpretation of the interview results. The first given quote from interview 2 additionally stresses that formalizability can be, precisely in the manner of Sfard's meta-discursive rules, a regulative for discursive decisions (Sfard, 2001, 26 f.).

Second, formalizability, when interpreted as a meta-discursive rule, can be understood as responsible both for the way and for the very possibility of successful communication (cf. Sfard, 2002, 31). The given quote from interview 2, expressing that formalizability of all accepted proofs is a "hygienic belief", and the second quote from interview 5, can be understood in this sense. Additional, socio-historical evidence corroborates this claim:

> I interpret formalization as a symbolically generalized communicative medium, which was developed precisely when, due to profound institutional changes, the prior social-integrative mechanisms became deficient. (Heintz, 2000, 252, my translation)

Still, formalizability does not fulfil all of Sfard's characterizing conditions of meta-discursive rules. This suggests that there are additional, competing as well as supplementing, epistemically relevant meta-discursive rules in actual mathematical practice. Some promising candidates can be read off the already presented empirical results, such as the possibility of perspective changes. Others need further empirical investigation, e.g., on the use and acceptance of informal meta-argumentation strategies.

CONCLUSIONS AND THOUGHTS TO THE FUTURE

The results of the study presented here, and also the sketch of meta-discursive rules as an alternative conceptual framework for their interpretation borrowed from research in mathematics education, have several implications for understanding the concept of proof in actual mathematical practice.

Induced by, but not limited to the investigation of formalizability, a general conceptual turn from proofs as mere linguistic entities to discursive proving actions was made in analyzing the true bearers of consistently explicable epistemic features. This highlights the relevance of conducting more detailed empirical studies of discursive practices regarding oral and written communication in actual mathematical practice, both in research and in university education. A subsequent examination of concrete implications for the teaching of mathematical proof could, as a first step, concentrate on university level mathematics education.

Nevertheless, the concept of meta-discursive rules can also serve as a connection between the presented research on proof and school mathematics. If the essential epistemic aspects of proof, with proof being one main putative component of a "nature of mathematics", are best understood as meta-discursive rules of actual mathematical practice[7], then teaching of the nature of mathematical proof should happen within proving discourses in classrooms, and include explicit reflections on meta-discursive rules in general, and formalizability as a special meta-discursive rule in particular. The validity of picture proofs, of the use of diagrams in proofs, and of argumentation assisted by geometrical representations are example topics for developing appropriate learning environments for such reflections on formalizability in, e.g., high school level maths courses.

In turn, such explicit reflections on meta-discursive rules in general and in particular should also become an integral part of pre-service and in-service teacher education.

NOTES

1. The study was embedded in a much broader agenda of developing a transdisciplinary approach of philosophical, sociological, psychological, historical and didactic research on actual mathematical practice that has been conducted by

members of the DFG scientific Network PhiMSAMP (Philosophy of Mathematics: Sociological Aspects and Mathematical Practice) 2006-2010 (cf. Löwe & Müller, 2010).

2. Note that since the works of Boole, Frege, Russell, Hilbert, and Gödel there exists a highly sophisticated discussion in (philosophy of) mathematics and logic on what kind of formal logic (e.g., regarding the order, and the meta-theory of the axiom system) is appropriate to formalize the foundations of mathematics (see, e.g., Hintikka, 1996, 2011). I am not concerned with this question here, but refer to today's most common foundation of mathematics within the first-order Zermelo-Fraenkel axiomatics (ZFC) of set theory. However, this assumption does neither essentially draw on the spectrum of semantics for "formalizable proof", nor on the essential features of formal proof like gaplessness and explicitness.

3. Possible bias was not taken into consideration here, as the interviewees were all professional mathematicians of high standing with arguably stable and grounded attitudes towards acceptable mathematical proofs in research.

4. Note that this is not to claim that such a conceptual turn is, or even should, be a necessary part of the aware image of mathematical proving held by professional mathematicians. Nevertheless, it is to claim that this turn should be part of any consistent, empirically informed philosophical consideration.

5. Azzouni (2004) coined the technical term of a "derivation indicator", but only for single arguments as linguistic entities, and with a different meaning.

6. As an example that illustrates the distinction, regard the following: "Investigate the function $f(x)=3x^3-2x$!" (see Sfard, 2000). An object-level discursive rule is that it is not allowed to divide the equation $0= 3x^3-2x$ by x to determine the nulls of f. But only meta-discursive rules determine what to do with f at all: "You are not sure whether you are supposed to list the properties of the graph (yet to be drawn!) or to admire its aesthetics; [...]; to make an investigation of the effects of real-life applications [...] or to check possibilities of transforming it, and so on." (Sfard 2000, 177)

7. Moreover, this approach offers interfaces to include a historical dimension. See, e.g., (Kjeldsen & Blomhoj, 2011) on historical case examples as a way to reflect on meta-discursive rules in school.

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