A CASE STUDY OF THE ENACTMENT OF PROOF TASKS IN HIGH SCHOOL GEOMETRY

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This paper describes how a Midwestern geometry teacher enacted proof tasks with respect to levels of cognitive demands. Data were collected via teacher interviews, observation protocol, audio, and video recording of the enacted lessons and teacher artifacts. The results suggest that the guidance offered during whole class discussion often reduced the level of cognitive demand of potentially richer tasks. Furthermore, whenever the teacher talked less and allowed students to work independently or in groups, the enacted proof tasks generally maintained higher-levels of cognitive demand.

Key words: proof, geometry, high school, teaching

INTRODUCTION

Proof plays various roles in mathematics, such as to verify, explain, discover, communicate, systemize or create an intellectual challenge (De Villiers, 1999). Within the classroom setting, to carefully unpack these roles of proof, teachers must seek to provide opportunities for students to engage in proving. According to Harel and Sowder (2007), "The essentiality of opportunity to learn must be recognized not only at the intended curriculum level but also in the teachers enacted curriculum"(p.827-828). Therefore, if we seek to increase students' opportunity to engage in proving, it is important to examine proof in both the intended and enacted curriculum. However, little is known about how proof is taught, in relation to curriculum materials which conveys the intended curriculum (Mariotti, 2006). Therefore, this paper sought to provide insight into how a geometry teacher enacts proof tasks by answering the following question: How does a geometry teacher enact proof tasks, with consideration to the levels of cognitive demands?

PERSPECTIVES

The manner in which teachers use curriculum materials can impact how proof is presented and what aspects of mathematical proofs are emphasized. Therefore, it is important to consider not only the curriculum materials used, but also how they are used during instruction. McCrone, Martin, Dindyal, and Wallace (2002) acknowledged that the four teachers in their study followed the textbook rather closely to structure the enacted lesson on proof, as well as for allocating homework assignments pertaining to proof, and used technology or hands on investigation activities sparingly. Since the mathematical content emphasized in textbooks can pose a challenge to teaching authentic proofs (Cirillo, 2009), it may not always be ideal for teachers to follow the curriculum rather closely. For example, Schoenfeld (1988) conducted a year long study of teaching and learning in a 10th grade geometry course. He found that, although the teacher exhibited "good teaching", the teacher's actions might have had a negative impact on students' perceptions of proofs. He suggested that the teacher's strict adherence to the curriculum might have caused students to differentiate between constructive and deductive geometry, consider the form of the mathematical argument to be paramount, and view doing proofs as a quick activity.

Bieda (2010) conducted one of the few studies that have examined curriculum materials during the enactment of proof-related tasks during instruction. Her results highlighted that when an opportunity to prove arose, students did not provide adequate justification approximately half of the time; and that 42% of the time teachers did not provide a response, 34% of the time teachers sanctioned students conjectures, and 24% of the time teachers requested the input of the class. She acknowledged that teachers were likely to provide positive feedback for non-proof arguments as if it were general arguments. Bieda concluded that "teachers in the classrooms observed did not provide sufficient feedback to sustain discussions about students' conjectures and/or justifications…[and] when a teacher provided feedback to students' justifications, it was not sufficient to establish standards for proof in a mathematics classroom" (Bieda, 2010, p. 377).

METHOD

This case study employed qualitative methods to investigate how a geometry teacher enacted proof tasks, with consideration to levels of cognitive demands (*memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics*). It is drawn from a doctoral dissertation, which examined how 3 geometry teachers use their geometry textbooks to teach proof. During the 2011 Fall Semester, I examined how a teacher used *McDougal Littell Geometry* (Larson, Boswell, Kanold, & Stiff, 2007) to facilitate students learning to prove. Before observing lessons pertinent to reasoning and proving (Chapter 2), parallel and perpendicular lines (Chapter 3), and congruent triangles (Chapter 4), I conducted a textbook analysis of task features and levels of cognitive demand of proof tasks, for the identified chapters. Of the 977 tasks analyzed, only 13.1% were proof tasks (tasks which explicitly required students to write a complete proof, or complete a skeletal proof such that the finish product illustrated a complete proof).

The Mathematical Task Framework (Henningsen & Stein, 1997; Smith & Stein, 1998), which defines levels of cognitive demand, was used to code proof tasks as written, planned and enacted. Proof tasks coded as *memorization* reflected skeletal proofs, in which students had to fill in the blank to complete a proof argument. *Procedures without connections* proof tasks included tasks that require matching statements, or are clones of examples provided in the chapter. Proof tasks that reflected *procedures with connections* included writing proof plans, or tasks that can utilize procedures to facilitate some degree of thinking. Such tasks can help students make connections between diagrams, postulates, and symbolic representations. Finally, proof tasks coded as *doing mathematics* required students to write a complete proof that was not similar to previous tasks and examples or is not algorithmic, and may change the context or utilize a different representation. Such tasks requires great depth of critical thinking, and facilitate students engaging in evaluating the merit or lack thereof for using a particular postulate to develop a proof argument.

To triangulate data, I utilized multiple data sources: interviews, physical artifacts, audio, and video recording of the enacted lessons and an observation protocol. The observation protocol used during the enacted lesson documented the classroom climate, instructional tools used, how the tasks were facilitated, levels of cognitive demand of the tasks, and proof schemes observed. Multiple researchers assisted with coding the written tasks. We had an inter-rater reliability agreement of 89%. Furthermore, an additional researcher accompanied me to more than 25% of the observed lessons. Our coding of the observed lessons on the observation protocol was generally consistent.

Participant

Purposeful sampling was used to identify the teacher studied. Mr. Walker (pseudonym), a fifth year teacher and head of the mathematics department, taught at a rural school, and used *McDougal Littell Geometry* (Larson et al., 2007) for at least three years. Mr. Walker obtained his undergraduate degree in Statistics, and subsequently obtained a Master’s degree in Mathematics Education. Additionally, he has taught high school geometry every year since he began teaching. Being one of the two mathematics teachers at the school, he taught introductory algebra, college credited algebra and statistics courses, and calculus. Mr. Walker believed that proof was needed in teaching mathematics because it fostered students gaining an appreciation for mathematics, and the work of mathematicians who contributed to the theorems and postulates that is visible within textbooks. He also believed that proof assists students to "become a little more logical in their areas of thought; not just math". Mr. Walker asserted that teachers' experience can influence how proof is taught, and acknowledge his preference for using the two-column proof representation. He stated, "…the two-column proofs are the easiest to see logical steps, so that is what I spend the most time teaching. Also, I may be more rigid in the steps that the students must show me... I don't like to find missing steps in logic according to our geometric postulates or theorem" (September 29, 2011- Follow up interview- sent via email).

In Mr. Walker's class students were first required to prove a theorem, before they could use it as supportive reasoning in a future proof. To facilitate students writing proofs, Mr. Walker required students to work in groups to construct proof arguments for proof on cards, or organize shuffle proof arguments to create logical arguments to exchange with other groups.

Mr. Walker acknowledged that the students whom he taught generally had a negative disposition towards proof tasks. He suggested that students' peers tell them that proofs are difficult, and therefore students are biased against proofs before entering the class. He also believed that students' negative disposition towards proofs were due in part to a lack of motivation to state their ideas with appropriate reasoning.

Furthermore, Mr. Walker was aware that proof can play multiple roles within a mathematics classroom, and suggested that the procedural nature of proof in geometry reduces the potential value of the proof.

He acknowledged that the textbook provided limited opportunity for students to engage in proving; hence, he often sought to provide supplementary proof tasks. Therefore, he was chosen as a unique case since he intentionally sought to increase the opportunity for students to engage in proving in his geometry course despite students' disposition towards proofs, and limited amount of proof in textbooks. Moreover, he was selected because he had a greater flexibility to progress through the textbook at his own rate, unlike the other two teachers studied for the dissertation who planned instructional activities and assessments with their geometry team. I observed Mr. Walker teach 75 minutes geometry lessons, 13 times, during the 2011 fall semester, in which he sought to expose students to proving. His high school geometry class consists of 9^{th} and 10^{th} grade students. The allocation of class time was devoted to reviewing solution to homework assignments, working on proof tasks in groups, and concluding lessons by providing solutions to assigned proof tasks.

RESULTS

Mr. Walker desired for his students to learn to reason effectively, and emphasized that the order matters in how a proof argument is presented. Supportive reasoning was emphasized for each step of the proof. He gave students a list of 28 reasons and regularly quizzed students about the content on the list. The list included definitions, properties of basic operations, properties of equality (such as reflexive and symmetric), theorems about congruent, and segment and angles postulates. The list of reasons had all of the necessary information to complete proof tasks that were commonly assessed. Hence, the list could be used to complete lower-level task rather quickly, and was used as a reference point for higher-level tasks. Thus, it could be argued that the list was an implicit form of teacher intervention, even when the teacher remained silent while students worked on proof tasks independently. Most of the proof tasks posed required six or fewer steps and used the two-column proof representation.

The textbook was used to assign homework, and structure the lesson. If he deviated from the textbook, the tasks he used aligned with the lesson objective of the textbook, and were meant to facilitate students learning how to prove. Mr. Walker frequently supplemented the textbook with additional proof tasks. The supplementary tasks posed increased opportunities for students to engage in higher cognitive thinking. Based on conversations with Mr. Walker, his deviation from the textbook was due to his desire to pose more higher-level cognitive demand tasks. He acknowledged that the textbook had limitations, and he tried to overcome them. Mr. Walker remarked, "I guess, there's just not enough like, if I look in this section in the book there's one, there's two proof of how we want them to be thinking about like" (November 3, 2011- Follow up interview at the end of the lesson). He also noted that sometimes the order in which content is presented in the book might not be logical, so his goal was to ensure the content progressed logically.

Although Mr. Walker's whole class instruction often reduced higher-level cognitive demand tasks to *memorization* or *procedures without connections*, when students worked in groups higher cognitive thinking was evident. An example of Mr. Walker reducing the level of cognitive demand of a proof task was visible on November 10, 2011, in which he required students to prove two triangles were congruent, which shared a common side. He said, "All right, I'll get you started" and proceeded to complete the proof in its entirety. In doing the proof he asked students to select one of the 6 theorems of congruency from the board to support the premise that the triangles were congruent. When a student selected an incorrect reason, the teacher continued by stating the correct reasoning and concluded the proof. Hence, the opportunity for students to engage in doing a higher-level cognitive proof task was not provided due to the excessive guidance provided by the teacher. Therefore, although the proof task had the potential to be considered a *procedures with connections*, if completed by students, when enacted by the teacher, the level of cognitive demand of the task was reduced. Mr. Walker did not pose any tasks that reflected *doing mathematics.* Although he had good intentions (which was to facilitate learning how to prove), the guidance offered during whole class discussion often reduced the level of cognitive demand of potentially richer tasks.

Cognitive Demand of Tasks during Mr. Walker's Enacted Lessons

Many of the tasks enacted in Mr. Walker's lessons were higher cognitive demand tasks. Table 1 indicates the level of cognitive demand of tasks for the original, planned, and engagement with the task during the enacted lessons as documented on the observation protocol. The original task depicted task as written, the planned task is the teacher's stated intention of how he intended to use the task during the lesson, and the engagement with the task is how the teacher actually used the task during the enacted lesson. In three lessons there existed multiple levels of cognitive demands for the various tasks posed. The shift from the original tasks to engagement with tasks suggests that when enacted the level of cognitive demand was reduced. It further suggests that half of the tasks Mr. Walker posed reflected *procedures with connections.*

Table 1. Levels of cognitive demands observed during 13 of Mr. Walker's geometry lessons.

Mathematical Tasks in Relations to the Levels of Cognitive Demands	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
Original Tasks	2	6	8	0
Planned Tasks	2	6	8	0
Engagement with the Tasks during the Enacted Lesson	2	8	6	0

Mr. Walker's *memorization* tasks often required students to restate postulate, theorems, and rules. He believed that, in order for students to prove, they must know a list of reasons. For example, he reminded students that the definition of angle bisector could be used to prove that, if an angle is bisected, the two angles formed are congruent. Mr. Walker said, “Good, definition of angle bisector. So this is on your list of 28 items. Basically what the definition of an angle bisector just says; it’s a ray, or a line, or a segment that divides and angle into two congruent triangles” (October 18, 2011- Enacted lesson). He readily referenced the list as a tool to identify appropriate reasoning to support claims made.

Writing statements about congruent triangles often were *procedures without connections*. He required students to place marking on the diagrams, identify corresponding sides and angles, solve equations, and draw diagrams. For example, Mr. Walker stated,

> We’ve got a lot of problems with segments and whenever we do a proof with segments, and we’re going to have to set an equation, there are usually two things that are going to help us set up an equation. With segments, it’s either that constant to midpoint or the segmented addition postulate. With angles, it’s the exact same thing except instead of, you usually have a midpoint of an angle but we’ve got angle bisectors so we could use an angle bisector to set up an equation or the angle addition postulate. So you’re going to have to look at the given information and kind of decide which of these can I use to set up an equation. Let’s keep that in mind. (September 8, 2011 - Enacted lesson)

Figure 1 shows an example of a proof he used to illustrate the procedure of using the segment addition postulate on September 8, 2011. This was categorized as a task of *procedures without connections*.

Figure 1. Mr. Walker’s proof tasks used to illustrate segment addition postulate.

Given: AC = BD

Prove: AB = CD

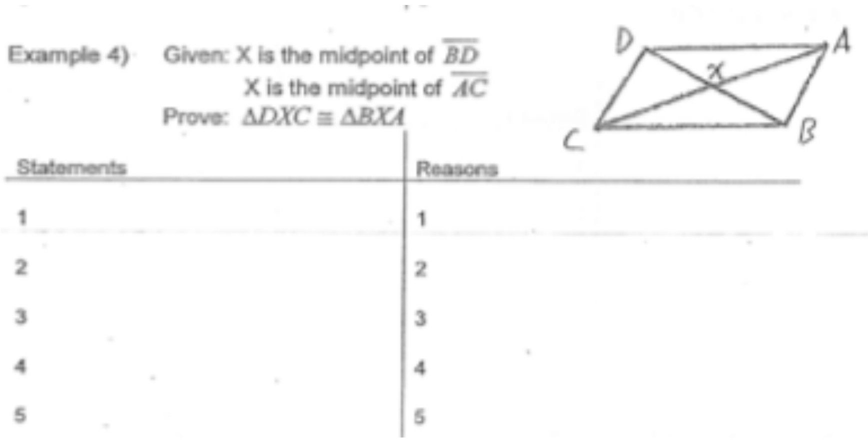
Proof

Statements	Reasons
1. AC = BD	1. Given
2. AB + BC = AC	2. Segment addition postulate
3. BC +CD= BD	3. Segment addition postulate
4. AB + BC = BC +CD	4. Substitution
5. AB = CD	5. Subtraction property

Among the tasks that Mr. Walker posed involving *procedures with connections*, include: asking students to write complete proof, organizing shuffled proof statements and reasons to make logical proof arguments, and assigning projects in which students had to construct a town that preserved the placement of buildings in relations to parallel and perpendicular lines, or write a story that logically links 10 conditional statements.

Admittedly, enacted tasks that reflected *procedures with connections*, the teacher was a silent participant in the group discussion. Based on my classroom observations, although students evidently engaged in higher-level thinking in their respective groups, during whole class discussion, the teacher merely provided the solution to the proof without requiring students to share how they constructed the proof. Figure 2, is an example of a proof task Mr. Walker wrote (November 15, 2011- Teacher artifact) to complement Section 4.6- Use congruent triangles).

Figure 2. Proof task Mr. Walker wrote that reflected *procedures with connections.*


Example 4) Given: X is the midpoint of $\overline{BD}$

X is the midpoint of $\overline{AC}$

Prove: $\Delta DXC \cong \Delta BXA$

Statements	Reasons
1	1
2	2
3	3
4	4
5	5

IMPLICATIONS

The results suggest that Mr. Walker provided excessive guidance when he discussed solutions to proof tasks. Excessive guidance is not ideal, since it can potentially limit the opportunity for students to write proofs independently and engage in discourse about their proofs. The teacher led discussion was merely to provide solutions to proof tasks rather than have students share their reasoning and possibly critique the reasoning of others. Although he provided the opportunity for students to engage with proof tasks during the enacted lessons, his whole-class discussion provided little opportunity for students to reflect on the merit of their arguments or other means to make the same conclusions. The weak questioning strategies employed during whole class discussion, which required generally recollection of facts, did not require students to reflect on or critique the reasoning employed in constructing the proofs. Such practices could potentially devalue the importance of proof, or hinder students from conceptualizing the validity of their mathematical arguments or developing mathematical habits of minds (Cuoco, Paul Goldenberg, & Mark, 2004). Although this study is not generalizable, since it focused on only one teacher from the Midwest region of the United States, it sheds light on how a teacher enactment of proof tasks can potentially reduced the level of cognitive demands of a task.

Requiring teachers to pose *procedures with connections* and *doing mathematics* proof tasks does not guarantee that students will engage with the tasks at the same level; considering that a teacher's actions during the enacted lesson can diminish the level of cognitive demands. Therefore, future researchers ought to examine roles teachers can play to ensure teachers' enactment of proof tasks maintains higher-levels of cognitive demands. Pre-service teacher programs, and in-service professional development need to encourage teachers to pose and enact proof tasks that require critical thinking. Hence, video recordings of effective and ineffective teaching of proof in geometry are needed, such that teachers can visualize practices that should be emulated, and avoided. The videos can provide opportunities to reflect on effective questioning strategies that can be employed and instructional strategies that can increase students' engagement with proof tasks.

Additionally, textbook developers ought to increase the number of proof tasks that requires higher-level thinking, in an effort to provide students more opportunity to prove. The guidance offered to teachers in teacher's edition of textbooks ought to promote the importance of having students prove, and should suggest strategies of how to unpack proof tasks to facilitate opportunities for students to engage in *doing mathematics*.

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