LOGIC, SOCIETY AND SCHOOL MATHEMATICS

David Kollosche

Universität Potsdam, Germany

An example of Aristotelian logic, which I show to be paradigmatic for Scholastic logic and school mathematics, is analysed for its social functions. This analysis builds upon a socio-historical interpretation of the meaning of early logic and shows its dialectic religious, epistemological and political dimensions: On the one side, logic can be used to appease, to emancipate and to amplify thought; on the other side, it might lead to fear, subjection and intellectual constraints. The combination of both allows logic and mathematics to become an instrument of power. Mathematics education, then, is the institution in which the acceptance of this instrument of power is cultivated.

Key-words: logic, Aristotle, religion, politics, epistemology

AN EXAMPLE OF ARISTOTELIAN LOGIC

Research on the social functions of school mathematics tries to answer the question in which dimensions society and school mathematics interact. While Ole Skovsmose, Roland Fischer, Philip Ullmann and others have contributed in answering this question, many issues have not yet been addressed. This paper is a contribution to a deeper elaboration on the nature of mathematics and its impact on society. One of the main characteristics of mathematics is logic. This paper examines the social function of logic in the mathematics of German secondary education, using an example of Aristotelian logic. The discussion will address questions on a historico-cultural as well on an educational level: *Which social functions did logic serve in ancient Greece? What social functions does it serve in contemporary school mathematics?*

The Greek philosopher Aristotle is considered the founder of logic since he compiled, formalised and analysed rules for speaking and thinking in the 4th cent. BC. In philosophy, his *oeuvre* is a milestone. His logic processes the transitions in speaking and thinking which ancient Greek has gone through and which is expressed in the work of his predecessors, such as Anaximander, Parmenides, Socrates and Plato. But what had caused these transitions; which purposes does the new kind of thinking serve? The socio-cultural analysis of the four laws of thought which Scholasticism has identified in the work of Aristotle (e.g. Schopenhauer, 1813/1903, § 33) will provide answers for these questions. The four laws of thought are:

1. Law of identity. *Everything stays the same, nothing changes.* The principle of identity is tautological as long as it is read descriptive. Read prescriptive, i.e. as the rule to speak and think in a way in which everything stays the same, the prin-

ciple of identity provides our speech and thought with concepts that are reliable in the sense that they do not change their nature with the speaker, the location or time. Already at the beginning of the 6th cent. BC, Anaximander of Miletus, probably a student of Thales, had argued for the existence of something “infinite” which is “indestructible”, “deathless” and “imperishable” (Aristotle, trans. 1930, 203b). Half a century later, Parmenides seized the idea, described it with similar adjectives and called it *truth* (*aleteía*; Parmenides, trans. 2009, pp. 14–23).

2. Law of excluded middle. *Everything is or is not; there is no other way.* The law of excluded middle divides our statements into two categories, e.g. truth and the false, and leaves no other option.
3. Law of excluded contradiction. *Nothing is and is not at once.* The law of excluded contradiction demands a decision between the categories. Together with the law of excluded middle, it forces the statements of speech and thought into an antagonism introduced by Parmenides: the antagonism of being and not being, of true and false, leaving no room for a state in between or beyond the extremes.
4. Law of sufficient reason. *Everything but one thing has a reason and is defined by it.* Distancing himself from mythological thought, Anaximander claimed that everything but one thing has a reason which is its destiny (Anaximander, trans. 2007, p. 35). Therefore, he is often considered the founder of scientific thought. The law of sufficient reason is a method to decide over true and false on the one hand, and provides a scheme with which to order ideas on the other hand.

RELEVANCE OF THE EXAMPLE

The four laws of thought of Aristotle are relevant for philosophy, mathematics and contemporary school mathematics. The discussion of syllogisms constitutes the core of Aristotelian logic. It addresses the question which forms of conclusion are certain. Certainty here means that the truth of a statement can be shown with necessity and out of sufficient reasons. As the laws of thought are essential for the discussion of the syllogisms, they form an essential part of Aristotelian logic. Certainly, the logic of Aristotle was not uncontroversial among his contemporaries: it competed against the old myths and against alternative philosophical schools such as the school of the Sophists represented by Protagoras. However, it was most successful as it attracted most attention in Ancient and later philosophy.

Euclid’s *Elements*, the first systematic compilation of mathematical concepts, theorems and proofs, and *the* reference of mathematics for two millennia, are strongly influenced by Aristotelian logic (Wußing, 2009, p. 191). This influence does not only consist of proofing as the method for the validation of mathematical statements. Aristotelian logic also influences the whole architecture of Euclidian mathematics: Its concepts are defined as tight as possible to avoid any variations of meaning and to provide them with unalterable identities; as sharp as possible to allow a clear inclusion or exclusion of every phenomenon; and as connected as necessary to be able to verify the properties of the concept. Today, mathematics has the very same architecture. Alternatives, such as polyvalued logics, logics without the law of excluded middle and others exist, but they are hardly used in mainstream or even school mathematics. Thus, school mathematics is based on Aristotle's logic:

- The concepts of contemporary school mathematics and their properties stay the same all through the curriculum. Usually, they are presented as universals which do not evolve or vary with culture or through history, leaving no room for an individual reading: Even and odd numbers, circles and so forth are its invariants.
- The concepts of contemporary school mathematics are structured in a way to either describe a phenomenon or not; they follow the antagonism of being and not being: A natural number is either even or odd; it cannot be both or something else. In fact, every classification follows this rationale: A straight line either shares points with a circle or not; and if so, then it shares either one point or two.
- The concepts of school mathematics refer back to their origin by which they are defined. Every super- and subordination, e.g. from the polygon down to the right-angled triangle, every implication, every transformation of terms and equations leans on the assumption that the truth of the new is already hidden in the original.

Although the relevance of the example of Aristotelian logic for the purpose of this paper has been illuminated, its social impact still has to be examined. The work of Klaus Heinrich, whose intention was to examine the "suppressed of philosophy" in its genesis (Heinrich, 1981, p. 10), provides a basis for a cultural analysis of early logic. He looks for radical changes in the thinking, feeling and activities of a society and asks for their reasons. His original perspective on Aristotelian logic is insightful as he succeeds in a fruitful combination of theory of science and cultural history. The following look at the birth of Ancient logic focuses the social circumstances that allowed Aristotelian logic to establish; it considers logic the intellectual manifestation of an experienced practice. What brings the thinkers of ancient Greece to cultivate that new form of thinking? Which were the social pitfalls and contradictions, the concerns, troubles and wishes that lead philosophers to logic?

LOGIC AND RELIGION

Before the birth of philosophy and science, only the myth offered explanations of the word, especially of its most alarming phenomena: the fatal threat of age and disaster. Hesiod's *Theogony* and Homer's epics describe the mythical world of gods who personify death, hunger, storms, floods, earthquakes, droughts and epidemics. These gods, and thus the phenomena they personify, were understandable as they were imagined as human-like creatures, who knew friends and foes, intrigues and murder.

The strongest influence on the nature of gods was assumed to be their descent: It is regarded as an unalterable trait, as a fate that cannot be escaped as in the curse of the house of Atreus (Heinrich, 1981, p. 99). The myth allows for a comprehensive orientation in the world; it is an early philosophy of nature. However, this orientation was threatened as democracy developed. The open discussions on the market place, the agora, did not only address politics but also philosophy, ethics and religion. Thus, traditional convictions were called into question, leading to a confusion of world views (Vernant, 1962/1982, p. 52). Socrates was the incarnation of scrutiny and died for it. The ongoing collapse of religious beliefs as well as political and economical structures led to a lack of orientation and security, and called for a new and possibly safer set of convictions.

However, Anaximander's disengagement from myth is only half-hearted. It is true that his world view goes without supernatural beings. But one the one hand, Anaximander sticks to the idea of the fateful power of descent as he claims that everything has a reason which is its destiny. This connection becomes clearer in the original word *archē*, which means not only reason or cause but also origin, descent and birth. On the other hand, Anaximander still believes in an ever-reliable existence, which he himself considers "divine" (Anaximander, trans. 2007, p. 37; Heinrich, 1981, pp. 60ff). Parmenides, in turn, presents his logic as a divine revelation. Calling the idea of the everlasting *truth*, he founds essentialist philosophy, which imagines a world of imperishable truths and dominates a large part of Antique philosophy (Vernant, 1962/1982, pp. 131f). In Heinrich's reading, the four laws of thought that had developed with Anaximander and Parmenides are an abstract derivative of the mythical religion – a world view that preserves the ideas of descent and the imperishable but avoids the debated existence of Hesiod's and Homer's gods. The spirituality of the laws of thoughts consists of the belief in the imperishable which opposes the threat of passing away and of the conviction that everything is connected and determined by ancestry. The law of identity is the imperative to belief in the imperishable. The laws of the excluded middle and excluded contradiction create the need for a decision between being and not being, between the true and the false. It is the mixtures and alternatives of being and not being, becoming and passing away, that are excluded. Therefore, Heinrich recognises early logic as a doctrine of salvation: "'Do not be afraid', for there is an existence which is not affected by fate and death" (my translation; Heinrich, 1981, pp. 45f).

LOGIC AND KNOWLEDGE

Apart from the promise of salvation, the presented example of logic offers a paradigm to order and explain the phenomena of our world. Its rules are easy and familiar; they provide a supportive and socially accepted 'machinery of thought' which allows thinking to approach complex fields on well-concerted ways. This potent form of thinking is the answer to the confusion caused by the collapse of traditional religious, moral and political beliefs and it is the impetus of philosophy. An illustration of the latter is Plato's Socrates who, in the dialogue with Theaetetus, stated that "suffering from confusion shows that you are a philosopher, since confusion is the only beginning *(archē)* of philosophy" (Plato, trans. 1921, pp. 155f; incorporating the translation in Heinrich, 1981, p. 31).

The price for the extended range of thought is the limitation of the thinkable. It is a main point in the *Dialectic of Enlightenment* by Horkheimer & Adorno that logical thinking loses sight of everything that does not fit into the antagonism of being and not being. Spinoza's logical ethics and the *Tractatus* of the young Wittgenstein, who wanted to trace language back to logic, show the limits of logical thought. Historical and cultural alternatives prove that logic in the form of the four laws presented is not the only form of fruitful thinking. Homer's epics, which count to the earliest records of ancient Greek writing, present meaning in analogies instead of logically ordered, in colourful images that connect the phenomena and emphasise commonalities. Even in the boom of early logic, Greek philosophy has critical schools such as the one of Heraclitus, who confronts the belief in the imperishable and true with the idea that everything is in flux and nothing persists, that "one cannot step twice into the same river" (Heraclitus, trans. 1979, p. 53). Today, many indigenous cultures understand and approach the world in a way which is based on the ideas of flux and mixtures instead of on the stasis of the unalterable and separated (Little Bear, 2002).

LOGIC AND POLITICS

Aristotelian logic also has a political dimension, which Xenophanes points at when he calls it a 'technique of reasonable speaking' (*logikē technē*): It is a tool for public speech, a tool for convincing and discrediting. Ancient Greek city states were usually governed by a democratically organised military aristocracy which represented a large part of the bourgeoisie. Politics was performed on the market place, where orators had to win majorities for their political campaigns. Politically powerful were those who performed best in convincing the people and discrediting the opponents (Vernant, 1962/1982, pp. 46–68). With his work, Aristotle wants to give directions for a convincing speech. He wants to show "what we must look for when refuting and establishing propositions" (Aristotle, trans. 2006, § 1). The question for truth and its origin, the rejection of the indefinite and unsteady, the exposure of inconsistencies, the installation of antagonistic options and the demand to decide are rhetoric weapons provided by philosophy. The teaching of such tools to the politically ambitious military aristocracy provided an income for many philosophers and explains their interest in logic. Thus, early logic constitutes a basis for democratic decision making. It provides socially accepted rules for argumentations in political discourses and helps to reduce more repressive forms of power. Nevertheless, even a democratically legitimated administration needs tools to control the population. Philosophy develops techniques which allow taking advantage of the logical laws of thought. As the acquisition of these techniques requires prosperity, logic has been an instrument of power for the aspiring middle class since its very beginning. As intended by Aristotle, the popular acceptance of the laws of thought subjects the masses to a form of speaking and thinking that they have nothing to set against.

THE DIALECTICS OF LOGIC

The cultural-historical analysis of the example of Aristotelian logic reveals its dialectic nature and allows for a comparison of the benefits and constraints it brings. On an epistemological level, it has already been pointed out that the laws of thoughts *expand* the range of thinking while *narrowing down* its focus. On a political level, early logic constitutes a *tool of power* which allows a less violent form of government but is *restricted* to one social class. Apart from that, the dialectics of logic become visible on a religious level: The belief in an imperishable truth might appease people who fear changes, especially the passing away. The emotionality with which Anaximander's infinite and Parmenides' truth is defended against any variation indicates a defence of this pacification. As Heinrich points out, even the *non* in Parmenides' "non being" (*mē einai*) translates with an undertone of menace (Heinrich, 1981, pp. 45f). Parmenides presents his logic explicitly as the salvation from an insane form of thinking – a form which "changes ways", which "considers being and not being the same and yet not the same" and which believes in "becoming and passing away". Parmenides' poem compares to a religious conversion using defamation and dictation: Those who think differently are said to be "double-headed", "helpless", "erratic", "drifting", "deaf" and "blind", "lost in confused wonder" and "unable to make decisions" (my translation; Parmenides, ed. 2009, pp. 17–23). The reader is told what to think and consider: to avoid the dissidents and follow logic.

However, the Frankfurt School points out that the belief in the unalterable only appeases some people while it frightens others: A world whose essence is invariant is a world that people cannot contribute to, that cannot be affected and must be endured. For Parmenides, humans are not the measure of all things (as Protagoras claimed to point out that everything is characterised by how we perceive and use it) but things *are by themselves*, beyond any human impact (Heinrich, 1981, pp. 32ff, 42). Consequently, the dead truths can only be approached in a quest for unveiling their dead and timeless mysteries. The dying Socrates believed in that when he told Phaedo that "the philosopher desires death" which is the ultimate "separation of soul and body" and frees the philosopher "from the dominion of bodily pleasures and of the senses, which are always perturbing his mental vision" and hindering him "to behold the light of truth" (Plato, trans. 1892, pp. 64–65). It is the paradox of essentialism that it forms an alliance with death to defend the passing away.

There is a touch of irony in the fact that logic cannot justify its basic laws in argumentation but has to build on myth and demand obedience as seen in the works discussed above. After his postulate of the law of excluded contradiction, i.e. that "it is impossible for anyone to suppose that the same thing is and is not", even Aristotle scolds: "Some, indeed, demand to have the law proved, but this is because they lack education; for it shows lack of education not to know of what we should require proof, and of what we should not" (Aristotle, trans. 1933, p. 1006).

Considering the actuality of the dialectics of logic, it has to be admitted that today such a pathetic statement concerning the laws of thought would hardly be expected. In Western society these laws are widely consolidated and accepted whereas in ancient Greek they had to be defended against alternatives. Right in the genesis of what came to be Aristotelian logic, in the time when the laws of thought had to fight for their place in the world, the social background of logic comes to light. Therefore, ancient culture can tell us about disputes whose consequences we have learnt to accept without question. Indeed, it can hardly be claimed that the cultural-historical analysis of the example of Aristotelian logic does not affect our age: On the one hand, the exclusion of the changing, of alternatives and mixtures as well as the orientation on ancestry are integral parts of Aristotelian logic, regardless of the social circumstances of the time; even today, they open a field of possibilities for religious, epistemological and political use. On the other hand, the fear of passing away, decision making in democracy and the organisation and validation of knowledge are still challenging our culture (although they might appear less urgent due to the techniques we already have developed to deal with them).

Logic and mathematics have been enormously influential for philosophy and science as we know it today. In the age of Enlightenment, when science had to find its meaning and place in society, logic and the empirical method were the points of orientation and left a formative imprint on modern thought. René Descartes was a pioneer of this process. He was convinced of "the great superiority in certitude of Arithmetic and Geometry to other sciences" and argued that "in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry" (Descartes, 1684/1990, pp. 224f). Mathematics is considered a prototype of science as it can be more logical than any other science: Its objects can be alienated from our world as far as necessary to fit the logical form and the validation of its assertions does not need any experiments but relies on logical argumentation alone. Mathematics has become modern not by turning towards empiricism but by petrifying its unique status. The foundational crisis of mathematics was connected to the critique on the idea of an unalterable truth. When the crisis was silenced by David Hilbert by negating all connections between mathematics and the world we live in, Hilbert installed mathematics as the science of pure structures: as a science that commits itself thoroughly to the order of logic (Hilbert, 1922).

LOGIC, MATHEMATICS EDUCATION AND SOCIETY

School mathematics is only a part of what is called 'mathematics' today – a part with a long tradition of specialised content and its presentation. As school mathematics builds on a long tradition and has little structural need for contact with contemporary academic research, it is not surprising that it hardly includes any new mathematics or any new philosophy of it. Especially the idea of 'truth' is more classical than modern, more Platonic than Constructivist. School mathematics excludes any contents that could threaten the classical idea of truth, e.g. non-Euclidean geometry, paradoxes of set theory or alternative logics. On the contrary, it adds only those parts of mathematics to its Euclidean core which approve the power of logic: Calculus and probability theory demonstrate how even the infinite and chance can be mastered by logic.

Experience shows that German students stop calling a task 'mathematical' when it no longer has a unique solution. Obviously, school mathematics follows the law of identity in allowing only one true and right answer, excluding any variation, individual interpretation and ambiguity. How school mathematics effects students can be traced in school books, e.g. in Brückner's (2008) representative school book for the 7th grade of high school. In there, we do not only find an overwhelming dominance of tasks that allow for only one true and right answer and tasks that ask for the "truth content" of statements or for a decision between "true or false", but also a task in which the reader is told that an algorithm "was applied wrongly". The fact, that the authors provided this task with the emphatic title "Caution, mistake! Watch out!" demonstrates how uncommon it is that the mathematics school book provides something else than 'the truth'. This taboo of the alterability of mathematics enables students to believe that school mathematics is about true and right answers and that, in general, every task that can be formulated mathematically has a distinct solution. Anything that could lead students to a different belief is excluded from the mathematics classroom. As school mathematics is the only mathematics students know, it prepares students to believe that mathematics is generally able to provide unique and unquestionable answers for any question. This turns mathematics into a tool of power with the help of which the public can be convinced of originally questionable decisions.

The same school book presents the following text without any further explanation, but with a sketch showing a circle and a line of each type (p. 142; my translation):

> Lines and circle can have different locational relations.
> Secant: a line that cuts a curve (g_1)
> Tangent: a line that touches a curve (g_2)
> Passant: a line that avoids a curve – the passing line (g_3)

Somebody not knowing these terms could rightly ask: *Can a tangent be a passant as it does not intersect but passes the circle undamaged? Can a secant be a tangent, i.e. can you cut without touching?* Only the reader familiar with the idea of classification could know that it is forbidden to place a line in more than one or in none of the three categories and that the definition quoted above is ‘meant like that’. Expecting the excluded middle and the excluded contradiction in the contents of mathematics is an *unspoken* prerequisite for its understanding. Thus, mathematics education does not only cultivate an unreflected and politically problematic idea of truth which gives power to those able to use mathematics; it also provides this power unequally. Those who understand the latent order of mathematics gain the possibility to perform well and become confident in the use of mathematics while those who do not understand its latent order are excluded from its power. Not talking about the logic architecture of mathematics leaves a large part of the students without any chance of understanding it and keeps the circle of mathematicians as exclusive as possible.

Although I presented only two examples for connections of school mathematics, logic and society, experiences from the classroom and students’ comments on mathematics education (Jahnke, 2004; Motzer, 2008) indicate further connections, drawing even closer to the dialectics of logic presented before. On the one side, there are students who dedicate themselves to mathematics passionately, who understand and use its order, and who are able to explain and argue more convincingly than others. On the other side, there are students who are frightened by mathematics, who consider it lifeless and unapproachable, who – in spite of great efforts – do not understand its order, and who must accept the explanations of the teacher and capable students unchallenged. In an extreme case, we can speak about two different types of students: some who enjoy mathematics and logic and use it to order and present their knowledge persuasively, and others who – anxious and confused – distance themselves from mathematics and trust in the guidance of the mathematically educated. This mechanism separates a logically emancipated elite from its subjects. Particularly, this mechanism influences the students’ relations to mathematics. It determines how mathematics is perceived and how people react to it. Research on mathematical world views shows that such a predisposition might indeed feature a view of mathematics as an inerrant decision-maker, the involvement in which is frightening and too demanding (Leder, Pehkonen, & Törner, 2002). In consequence of this mechanism, mathematics serves as an instrument of power which is trusted by the majority of people and whose rationale is no longer questioned. Mathematics education acts as an institution which unconsciously installs this instrument of power. This study show that the social impact of mathematics education is not a matter of the style of teaching but connected to mathematics itself; for – since its very beginning – mathematics was the manifestation of a tool of power called ‘logical thought’. The question on how to face this social impact of mathematics is open to debate.

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