HISTORY OF MATHEMATICS AS AN INSPIRATION FOR EDUCATIONAL DESIGN

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Use of historical and cultural perspectives in education supports the development of mathematical concepts which are thus not as usually based on logical relations. They allow embracing new individual contexts of experience as well as methods of science and humanities. The use of models of historical development and the need of understanding of phenomena that foster or block learning challenges teachers as well as learners. It will be shown at the hand of concrete and relevant examples how historically, culturally and socially inspired problems can encourage an alternative approach to well-known mathematical concepts and deepen understanding.

INTRODUCTION

The employment of historical examples can enrich mathematical teaching in various ways. From the viewpoint of a historian, problems of inadequate methodology and a lack of contextual understanding might easily occur when historical material is used in the classroom. In this paper we do not put stress on the professional handling of historical material. We would like rather to consider the use of historical material as a source of inspiration in educational design and as a diagnostic mean.

In the first part we discuss different approaches to the employment of historical content at the hand of existing forms of presentation of historical material in teaching. Hereby we pay attention to the representation of the historical material and its potential to initiate in the class room a discussion of the socio-cultural aspects connected to mathematical topics. The second part will exemplify the relations of various models of historical development and will make related positions in learning theories explicit. This constitutes the basis for the conceptual understanding of historical aspects by means of mathematical awareness in the third part. Here we are guided by examples from mathematics textbooks and examples from the education of pre-service mathematics teachers for grammar school.

FORMS OF EMPLOYMENT OF HISTORICAL CONTENTS IN TEACHING

The use of historical material in educational design has a long tradition. For instance Walther Lietzmann gave in 1921 a lecture course on recreational mathematics at the University in Göttingen and emphasized the important role of social and cultural aspects as well as the potential of historical contexts for the design of mathematics instruction. He encouraged the publication of corresponding materials with direct use for teaching. Although Lietzmann gave in (Lietzmann 1922) extensive references, the discussion of the authenticity of the presentation of historical content (Fried 2001) was not a predominant issue. In Lietzmann’s book impulses to reflect about mathematics and incentives to deal with mathematical problems are given by means of varied, witty and curious comments. In contrast to the approach of a historian, where the existence and understanding of later mathematics is assumed, and is even part of the background, Lietzmann's approach does not involve an understanding of modern mathematics. Historical contents and examples are used to make other points of view accessible to the reader, to free her from thinking routines by the use of metaphors, unexpected representations, and confrontations in order to enhance a deeper understanding and ludic handling of mathematical objects.

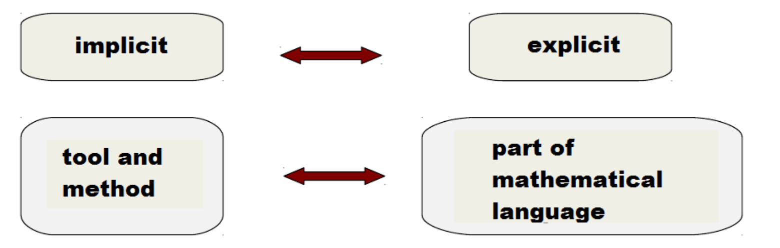


Figure 1: Dynamics of the development of mathematical concepts.

There are various models of historically oriented mathematics teaching developed by historically interested mathematics educators (see Fauvel & Van Maanen 2000, Jankvist 2009), however, for Lietzmann history of mathematics is a source of inspiration for storytelling. In most current mathematics textbooks we can find some historical content. The presentation forms range from historical anecdotes to excerpts from historical documents (Kronfellner in (Kronfellner 1998) gives an overview of commonly used forms of historical content in the classroom). These materials are usually designed as insets or appendices that fit in the textbook design. They are additions to the canonical representations in the given textbook. The related omissions and circumlocutions thus easily lead to erroneous ideas about the historical development of mathematical ideas. In developing a socio-cultural context of the historical material the teacher should be aware of adjustments that have possibly been made.

Many textbook examples of historical inserts in mathematics textbooks show the effort to bring authenticity with learning and reading habits in line. The language and symbolism used correspond only to a small extent with those in the original historical sources. The presentations often build on the student's presumed skills and routines. The student is more likely to reproduce logical steps in the modern style than that she is stimulated to think in the framework of the historical context.

In the design of learning environments based on historical sources the deliberate inclusion of national diversity in language and culture that can be found in the classroom can often be rewarding. An example would be the inclusion of original texts. Another practical link between the mathematical content and Islamic culture is shown in (Moyon 2011), by attributing geometric problems of area decomposition to inheritance rules in the form of rules for splitting acreage.

Admittedly, a playful handling of historical events, oriented at imagination, transfer and variation is in danger of losing its historical authenticity but it can stimulate a much broader mathematical activity. Everyday knowledge and intuitive ideas can contribute amenable to different perspectives. When you want to understand historical hypotheses related to the constellations of celestial bodies (Jahnke and Wambach 2011), unwanted references to reality and modern general knowledge can interfere with the unbiased view. The latter interference can be avoided by an initial settlement of the problem in a fantasy world. The necessary alienation of everyday experience of space and time can be generated playfully by the transition into computer games, science fiction and fantasy-inspired worlds. The resulting relationships between socio-cultural and extracurricular mathematical approaches allow diverse developments. Versions of the transmission of mathematical problems in other worlds (in the literature there are many examples, e.g. Jules Verne, Kurd Laßwitz, Ian Stewart, Stanisław Lem, Terry Pratchett), and social relations in the world of mathematical objects (Isaac Asimov, Edwin A. Abbott) allow direct involvement of dramaturgic tools to deal with interaction of cultural and historical mathematical and scientific phenomena. Taking together these and many other examples, we can see a wide variety of possible inspiration from historical material in the mathematics classroom.

HISTORICAL PERSPECTIVES ON MATHEMATICAL DEVELOPMENT AND CONCEPT DEVELOPMENT IN MATHEMATICS EDUCATION

The references and school examples listed in the first section show that historical materials in textbooks usually occur as a reference to historical sources or as an illustration by means of a historical record of the mathematical concepts, objects, methods, or mathematicians just treated in the class. History here is strongly identified with the existence of historical sources, usually without paying attention to the historical source itself. This may partly be due to the fact that the use of historical sources for general goals of mathematics education, such as to experience mathematics as a living, evolving science or to create access to the cultural heritage, are difficult to interpret in such a general formulation. Thus the tool is turned easily into the goal and the historical work is reduced to mere reference to the existence of historical sources. The indeterminacy inherent in the historical approach can be

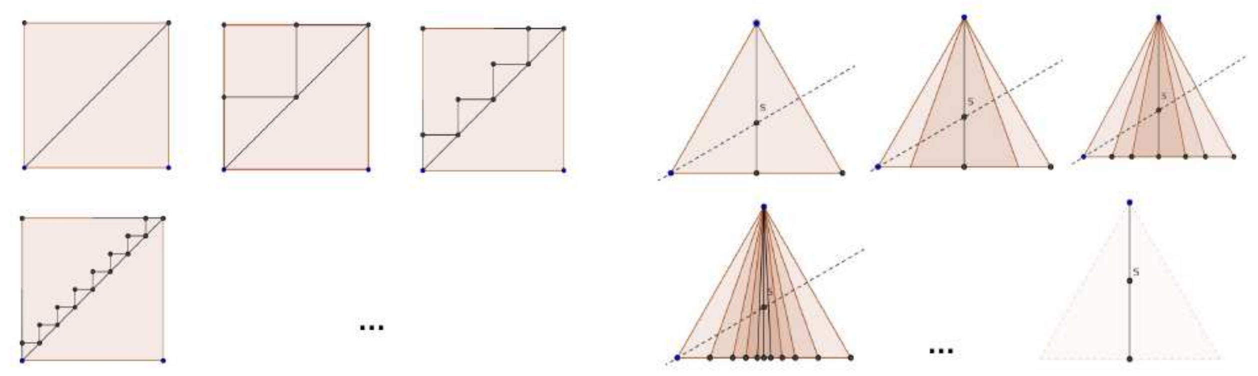


Figure 2: A mistaken approximation of the length of the diagonal of a square. (left) A mistaken determination of the center of gravity of a segment as the limit of the centers of gravity of a family of triangles. (right)

ignored because there is the simple control feature *availability of the historical source*. The exciting and very challenging task of designing a learning environment based on the historical source cited in the textbook by introducing historical contexts rich with potential for socio-cultural, mathematical and scientific development lies in the hands of the teacher. How can, however, a short section of a historical text be turned into a mathematical development and what can actually be understood under mathematical development?

In designing such learning environment the question of the underlying concept of a scientific theory, the choice of the general philosophy and of the appropriate developmental model plays a crucial role: is the historical progress based on endogenous and exogenous factors; is it based on the activities of some eminent personalities, or are there rather economic, political and social factors that contribute to the development of this subject? Is it necessary, for an appropriate understanding of the development of mathematics, to turn to methods such as source-analysis and source-interpretation, ethno-mathematical approaches or other perspectives?

For a historian in the choice of the developmental model the scientific criteria are in the foreground. When contextualizing a historical fragment, the distinction between perspectives on history as a tool and as a goal and further differentiation of these categories as introduced by Jankvist (2009) seems natural and fruitful. In the development of learning environments by means of historical sources the entertaining potential can also play a role. Whiggish or present-centered representations (for this distinction and for a discussion of their unscientific nature see (Wilson et al 1988)) can still inspire genetic instruction or, as the first part exemplified, by means of relocating the actions, personalizing circumstances, and introducing other ways of alienation we can trigger a change of perspective leading to deeper understanding. The mathematics educator, not bound by the norms of scientific rigor of the historian, is open to other developmental models and many more possibilities of the use of historical sources in developing teaching environments.

In a class, the concern is less on the development of mathematics as such as on the development of specific mathematical notions and concepts.

For the development of mathematical concepts in the classroom an important role is played by the implicit phase in which a mathematical concept is not systematized, defined or referred to, which phase is for the historian hard to grasp and not often studied. As appropriate structures or regularities at this stage, however, often appear in the form of a problem solving method or a representation not really anchored within the context of the language of mathematics, the implicit phase can be helpful in instructional design in motivating a definition or in helping to formulate a problem aimed at introducing a concept.

From a historical perspective, there are helpful approaches to extend intension-extension of the conceptual dynamics, such as in Hans Wußings model by *ostension*

(Scholz 2010). The presentation of the development of Euler’s formula, as Imre Lakatos in (Lakatos 1976) has reconstructed it, is oriented before all on the development of mathematics as a language. This developmental model may be whiggish for a historian. However, it makes important issues of epistemological debates of the last century accessible to a class discussion and it makes tangible the otherwise hardly conveyable idea that even the meanings of mathematical concepts are negotiated. For the long-term conceptual development in the classroom it is worth to think about what historical or other socio-cultural developmental models reflect the dynamics described in Figure 1. Even for local ordering and linking of concepts in school mathematics it is useful to deal with regularities in the development of mathematical language and with regularities in the development and formalization of mathematical methods from a historical perspective (Kvasz 2008).

DESIGNING LEARNING ENVIRONMENTS BY THE AID OF HISTORICAL PERSPECTIVES AND MATHEMATICAL AWARENESS

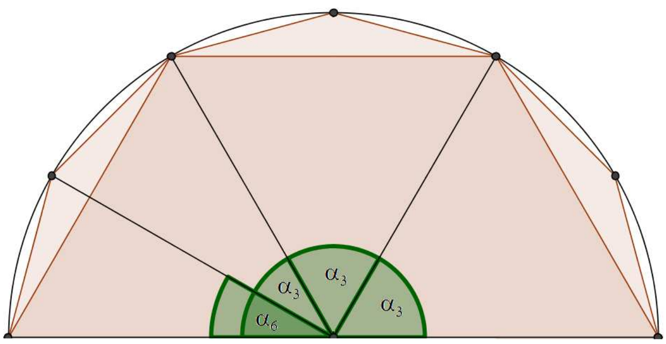
The development of mathematical concepts in the classroom is based on omissions, substitutions, rearrangement distortions, misrepresentations and other customizations transformations of relevant historical processes. On the other hand, there is not the one historical process of a concepts development but many aspects and perspectives under which one can see development as already discussed in the previous chapter.

The following example (Figure 3) shows that historical jackets of mathematical problems do not necessarily initiate a change in routine approaches or standard solutions. In teacher training seminars the solutions students had for this text book problem were hardly related to elementary or historical approaches.

These solutions, with only minor variations occurred in multiple (parallel) seminars

Approximation of π

The determination of the number π was in the history of mathematics an important task. Various processes have been developed. One method is to approximate a semicircular arc of radius 1 by a line of equal chords.



a) Calculate π for approximately three equal sections. Evaluate the results.

b) Develop a process that allows you to find a better approximation of π.

Figure 3: Example of an exercise in historical jacket.

on mathematics education. The tasks were part of a seminar presentation prepared by students on real numbers. The student's answer was to express half the length of the edges of half a regular n-gon by the sine of the angle π/n. They expressed their calculation using the calculator and multiplied this length by n/2 also with a calculator. The presentation of the solution consisted of giving symbolic expressions. The irrationality of the terms of the sequence, which approximate π was not noticed because the computer automatically rounded. Reflections on the convergence of the sequence and the transition from geometric objects (lines) to arithmetic sequences, i.e. sequences of numbers were not made.

Despite the historical jacket and the instruction for a direct calculation, the students calculated the solution with calculator using the sine function even for half a hexagon – although the three equilateral triangles give the approximation of π by 3 accurately. If you want to combine the cognitive process with the historical one, the reason for the superficial solution can be seen as a lack of *experimental* and *intuitive* awareness (see Kaenders and Kvasz 2011). The easy change between the geometric and arithmetic representation was based less likely to automation or a deeper understanding of the limit, but on the lack of experience with pathologies. The latter historically led to the necessity of the precise formulation of the limit concept. The two examples in Figure 2 (Lietzmann 1949) above do not represent this historical development of a precise formulation of a geometric limit. However, they show that they are necessary and can therefore lead to doubts about the solution and give rise to rethink the solution.

Simplifications of the long historical development of the concept of limit in the classroom are of course essential. Omission of pathologies in this example could be seen in relation to the historical development as an aggravating, trivializing simplification rather than a support to conceptual understanding.

The next example relates to a change in the usual representation of the solution of systems of equations. Given a system of linear equations with two equations and two unknowns, such as:

$$2x + 2y = 3$$

$$-5x + 2y = 2.$$

The approach proposed by the math textbook begins with the algebraic solution of the system: by equivalent transformations, the two equations can be converted into two equations, from which the solution $x = 1/7$, $y = 19/14$ can directly be read of. The corresponding geometrical solution process is started with the visualization of a given system of equations. Here, the linear equations can be brought into a form from which the transition *equation* ↔ *geometric object* is routine operation.

Since this has been trained in the context of drawing graphs of linear functions, it is a common textbook exercise to find the equation of the form y = mx + b, when the line in a coordinate system is graphically given. Just examples of the form x = b cannot be selected. In the next step, the transition between the expression y = mx + b, and the function graph is automatized. The visualization of the algebraic method or the "geometric solving" of the system of equations is now to determine the two function expressions corresponding to the initial equations, to draw the graph and read of the coordinates of their intersection.

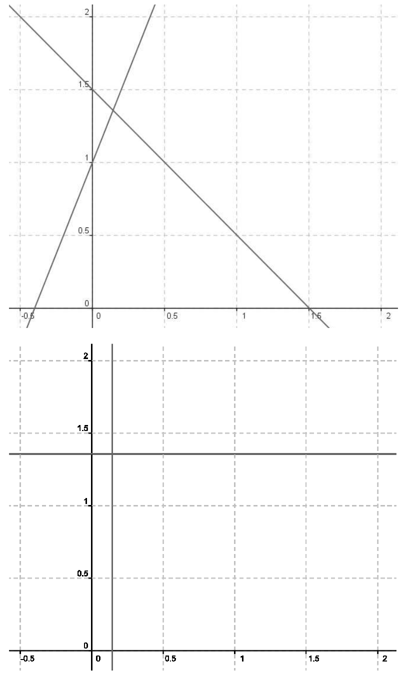


Figure 4: Visualizing of equivalent systems of equations:

$2x + 2y = 3$	⇔	$x = 1/7$
$-5x + 2y = 2$		$y = 19/14$

The lower situation does not exist in the language of school books.

The use of the concepts linear function, with slope and intercept, the ability to draw straight lines and the associated determination of the intersection of two graphs of functions by reading of the value of a function at a point are required as appropriate skills. The concept of equivalence transformations – in this case to determine the nicest representatives $x = x_0$, $y = y_0$ from the set of all pairs of linear equations with the solution set $\{(x_0, y_0)\}$ is not evident in the described approach: in the geometric representation of the equivalent transformations in each step the two lines preserve the intersection of the lines. Since the straight line $x = x_0$ is not common in the understanding of function graphs, the geometric representation of the algebraic solution method is reduced to the visualization of the given lines, and reading of the intersection. The described way of visualizing the equations, graphs is probably chosen also because the understanding of analytical geometry of curves in the plane, especially of lines is not developed.

The concept illustrated in Figure 4 of equivalent transformations and conservation laws, which is essential for the solving of systems of algebraic equation of higher order, and appropriate geometric representations by hyperplanes cannot be generalized by the transition to function graphs and values of a function at a point.

The language used in the example shows above all a lack of contextual awareness. Instead of a change of representation a visualization of an equation is performed. From this we can conclude a lack of logical awareness, too, i.e. the role of systems of equations in an adequate development of theory is ignored. When we orient ourselves in the historical process, then the later and in a different context developed representation of functions as graphs of functions in Cartesian coordinate systems is rather misleading here – the Cartesian coordinates go back to René Descartes (1637) and the concept of a coordinate change to Christiaan Huygens (1656). One could denote the misplaced use and the transfer of concepts that are developed in other contexts as *ahistorical implant*.

Another example deals with the currently conventional introduction of integral calculus, where three different aspects of the integral are introduced simultaneously:

- Integral as oriented area,
- Integration as anti-derivative,
- Integration as a way to determine a function from a given function of change.

One aim of this introduction is the motivation and direct introduction of the fundamental theorem of differential and integral calculus. To allow a direct connection between calculation of area and the determination of a function from the function of change, the change function is replaced by a step function, and thus the oriented area under the curve is replaced by the area of rectangles. The original functions are monotone, the determination of the step function is carried out graphically by estimating the areas of respective triangles. The replacement of the function by a step function aims at simplicity of calculation, but does not fit into the framework of the developed mathematical language of calculus: The geometric transition from secant to tangent with slope or the notion of actual speed would suggest an approximation with piecewise linear (rather than piecewise constant) functions. A historical insert of the parabolic segment method of Archimedes would show that it is a technically challenging problem and so, it could motivate the need for the later introduced Riemann sums.

The language used shows *experimental* and *intuitive* awareness. Upon further conceptual and technical development of the concept of the integral taken simplifications can nevertheless lead to motivation and understanding of issues and the development of logical and theoretical awareness counteract. The summary described the technically difficult problem of the definition of the Riemann integral, and of conceptual understanding of the relationship between differential and integral calculus will not do justice to the complexity of the task, so you might call the representation as a caring shortening.

A well-known transformation of historical contexts is what Freudenthal formulated as a criticism to the New Math movement *(anti-didactic) inversion*, which even today often determines the representation of higher mathematics at the University:

> "… the final result of the developmental process is chosen as the starting point for the logical structure in order to finish deductively at the start of the development. This genetic-logical inversion expresses itself as a didactical – or rather antididactical – inversion." Hans Freudenthal, 1991, Chapter 11, S. 305.

In this kind of concept development the focus is on the long-term development of *logical* and *theoretical* awareness. The inclusion of an implicit history of the development of a concept and the *ostension* of a notion would lead to a more balanced mathematical awareness that also includes *intuitive* awareness.

And finally, we introduce the well-known parable of Achilles and the tortoise by Zeno of Elea (495 - 430 BC.) Still today it is treated in many math textbooks. Unfortunately the Zenon paradox causes rarely the expected confusion or amazement. That may be partly due to the role of the problem as an application to the convergence of geometric series, which might inhibit a direct confrontation with the verbal formulation of the paradox.

It was interesting to observe that the arguments in the teaching seminar consisted in the verbal recapitulation of the steps. However, to find own formulations and phrases related to the initial formulation of the Zenon paradox were a problem to our teacher-students. Paradoxes force a change of perspective, re-orientation and provoke cognitive conflicts. One way to achieve this would be by the following possible question of Thomson (1954): Suppose a referee actuated whenever Achilles catches up to the new position of the turtle, the switch on a lamp. If the lamp is switched on or off when Achilles overtakes the tortoise? Also indirectly present ideas, such as the overall presence and familiarity with real numbers and their completeness, separate the students from the original and the paradoxical phenomena. The neglect of such indirectly involved factors when considering historical episodes can be called *cultural alienation*.

For the analysis of the concept development of the textbook examples, it was helpful to further investigate, identify and name the changes made to the historical development of concepts (e.g., reductions, simplifications, omissions, ...). To draw attention to several emerging problems in the integration of historical content, we have introduced these following terms: *(ahistorical) implant*, *(caring) abridging*, *(trivializing) simplification*, *(anti-didactic) inversion* and *(cultural) alienation*. For the previous investigation of handling and development of concepts in teaching at the hand of concrete textbook examples and the description of possible problems of precise formulation, of conceptualization and transfer the perspective of mathematical awareness was useful. Based on an analysis of the mathematical language, we examined two aspects: First, what qualities of mathematical awareness do not occur in the language and symbolism and second, by which evolution of the subject could these qualities of mathematical awareness be developed.

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