A FOCUS ON HIDDEN KNOWLEDGE IN MATHEMATICS THE CASE OF “ENUMERATION” AND THE EXAMPLE OF THE IMPORTANCE OF DEALING WITH LISTS

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The aim of this paper is to bring some elements (based on French research works) for the discussion. The curriculum of French preschool is briefly presented as well as the context of a French research which underlines that some kinds of knowledge are not identified as such but are required for the learning of counting in preschool (and also in other levels of mathematics learning). The knowledge “enumeration” is brought to light and defined (Briand, 1993). Another kind of situation is also developed regarding the construction of lists and their reading (Loubet & Salin, 1999-2000). The conclusion opens on the international perspectives of these kinds of results.

Keywords: enumeration, counting, lists, curriculum, measurement.

THE FRENCH CURRICULUM OF PRESCHOOL

This curriculum (BO - 2008, June, p. 12) applies to children from 3 to 5 years old. It is split into six parts: discovering language; discovering writing; becoming a pupil; physical development; knowledge and understanding of the world; creative development. Mathematics are integrated in the wide field “Knowledge and understanding of the world” which contains the following components: discovering the objects, discovering the matter, discovering the living, discovering the shapes and the magnitudes, approaching quantities and numbers, finding one’s bearing in time and space. The main advantage of such an approach is to promote an interdisciplinarity that avoids the compartmentalization of disciplines at school. The inconvenience lies in the fact that there are no clear directions for the teachers, who have to construct an organized teaching of mathematics. In this curriculum, there is no prescription for the didactical approaches that should be used in the classrooms. Moreover, some mathematical terms are vague. That is the case of the question of “classification”, for instance. This knowledge is dispersed in several paragraphs of the curriculum (of preschool and primary school): the mathematical words (to classify, to sort, to order, to list, to define, etc.) are often neither used nor linked in a correct manner (but that would require another article focused on the topic).

CONTEXT OF A FRENCH RESEARCH

Sometimes, children require some specific hidden knowledge (i.e. not taught and not identified as such in the curricula) in order to learn an identified knowledge of the curriculum (BO - 2008, June). The teacher cannot take it into account in his teaching either because this knowledge is diffuse or because it constitutes a kind of metaknowledge. It is very difficult for a teacher to grasp a knowledge which is not an identified knowledge in mathematics, and to connect it to some other taught kinds of knowledge; that raises the question of pre-service and in-service teacher training and the question of the components of the curricula.

The main question for a didactical research is twofold: what is precisely this hidden knowledge? How can one design didactical situations to develop this knowledge? Researchers have dealt with these questions in France for a long time (in the eighties, their results were published in the nineties and are now revisited), regarding the field of reasoning, of geometry (Berthelot & Salin, 1992), of logic (Loubet & Salin, 1999-2000), of the construction of the first numbers as well as for arithmetical operations or combinatorics (Briand, 1999 for instance; Margolinas, 2012). The main theoretical background of this research is the Theory of Didactical Situations (Brousseau, 1997), which is made up of two main steps: finding problems where the knowledge aimed at is the best tool for the resolution of the problems, and, starting from such problems, designing (a)didactical situations [1] suitable for students of a given level. Such a design uses several theoretical tools, mainly the notions of devolution of the problem [2], didactical variables [3], and institutionalization of knowledge and processes [4].

Brousseau (1984) and Briand (1993) have underscored several preliminary steps in the learning of counting. One of them is the notion of "collection". Indeed, in order to count, a pupil should perceive the object "collection" in order to measure it. To determine a collection brings a measurable space from a mathematical point of view; it requires several skills for a pupil including among others "pointing" (i.e. the act by which one gives a sign to an object but without structuring the set of objects; it can be the finger-pointing) and "denominating" (i.e. the act by which one gives a name to each object; it implies a structure of the collection). In the various situations designed by French researchers involving collections and actions on them, the first ones can deal with lists (to point a collection for instance, but also to match a picture and an object and then to elaborate a code to point several objects, and to represent the structure of a collection).

This paper will deal with two kinds of fundamental knowledge prior to counting: to enumerate and to deal with lists. What enumeration is will be explained first and will be followed by starting situations dealing with lists (a wider set of didactical situations is available in Loubet & Salin, 1999-2000 and will be discussed during the Congress).

ENUMERATION: A CRUCIAL KNOWLEDGE

A hidden knowledge which is required for counting

Several research works deal with the construction of the concept of number, exploring the manipulation of collections in particular. I will not develop here the results of Fuson et al. (1982) regarding children's elaboration of number word sequences and the five counting principles of Gelman & Gallistel (1978) (the one-

one principle; the stable-order principle; the cardinal principle; the abstraction principle; the order-irrelevance principle). I will highlight a hidden knowledge which is required for counting.

The following situation comes from a wider study (50 pupils, 6 years old – see Briand, 1993; this kind of study is now revisited by Margolinas (2012) who underscores the importance of the results of Briand and proves that these results also overlap the didactic of French). The pupils have a sheet where trees are drawn (Figure 1). The teacher asks the number of trees. The pupils can draw on the sheet, as they want.

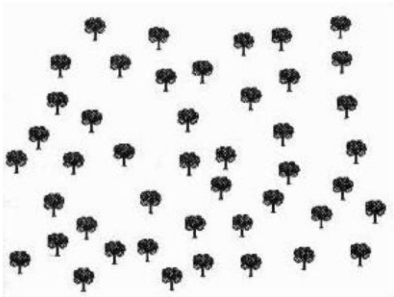

Figure 1 – Trees (in this case, the pupils have to structure the collection in order to count the trees)

In order to succeed, the pupils have to take the following steps:

1) *To be able to distinguish two different elements of a given set.*
2) *To choose an element of a collection.*
3) To say a number word (which is "one" or the successor of the previous number word in the continuation of "words-number").
4) *To keep in mind the set of the chosen elements.*
5) *To identify the set of elements which are not yet chosen.*
6) *To start again (with the collection of the elements that have not been chosen yet) 2-3-4-5, as long as the collection of the objects which are to be chosen is not empty.*
7) *To know when one has chosen the last element.*
8) To say the last number word.

In this series of actions required for counting, only the steps in straight (steps 3 and 8) refer to the numerical sequence. What Briand (1993) calls "inventory" is a task represented by the steps in italics: the pupils have to look over all the elements of a finite collection one time and only one. This task characterizes a non-taught knowledge, called by Briand "enumeration". Learning enumeration is not a part of teaching. It is a paradox that "enumeration" is not considered as a fundamental component of counting up activities or measurement of discrete quantities (for the construction of first numbers, as well as for arithmetical operations or combinatorics). The research of Briand (1993) (following the research of Brousseau, 1984) proves the necessity of activities involving “enumeration” in the pre-numerical domain. The following excerpts underline its existence.

Analysis of some strategies: the lack of the “enumeration” knowledge

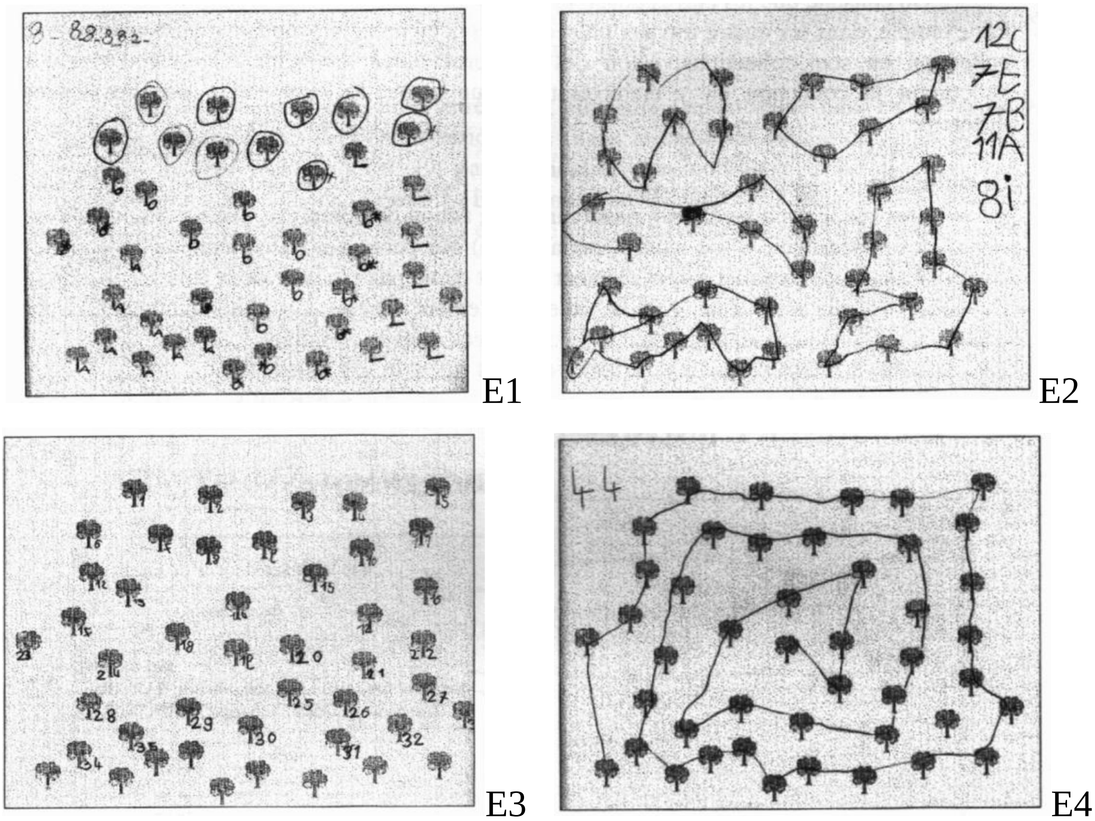

Figure 2 – Four processes (pupils are 6 years old)

Several strategies can emerge (see Figure 2). All of them need an organization of the space in order to take into consideration all the elements of the collection, without forgetting one of them. The lack of such a spatial organization implies difficulties in counting.

The first two pupils structure the collection into subcollections.

> E1: this pupil builds subsets and counts trees of each subset with a specific marking. A partition of the whole set of the trees is done. He builds the writing ‘8 8 8 8 8 2’. He misses the tree with a circle and a cross.
>
> E2: some trees are linked and then constitute a subset. A partition of the whole set of the trees is done. A letter represents each subset.

The other two pupils use an order to act on the collection.

> E3: this pupil explores the collection by line. The numbers are written but the pupil stops her/his counting (at 35) because there is no more line and the pupil

“does not know” where 36 is or where she/he can mark 36 (although this pupil can count way after 36).

E4: this pupil draws a path like a snail that will facilitate the counting. She/he finds 44 because she/he has counted the number of jumps and not the number of trees.

These excerpts show that the “enumeration” appears as a crucial knowledge. One can define the verb “to enumerate” with the following elements (see Briand, 1993, § 1.4 to 1.7 of chapter 1):

To enumerate: this is an action that involves taking into consideration each element of a collection, one and only one time. This verb refers to numbers from an etymological point of view, even if this action does not require the knowledge of the numbers. But the action “to enumerate” is a necessary action to count a collection. The enumeration implies the mobilization of several kinds of mathematical knowledge (from the exploration of space to combinatorics): it depends on the nature of the collection (if the objects are visible or if the objects are defined by properties).

A fundamental situation for enumeration

Briand (1993) has designed a fundamental situation for enumeration, for pupils between 4 and 5 years old. A didactical analysis of this situation will be developed during the working group with the help of videos.

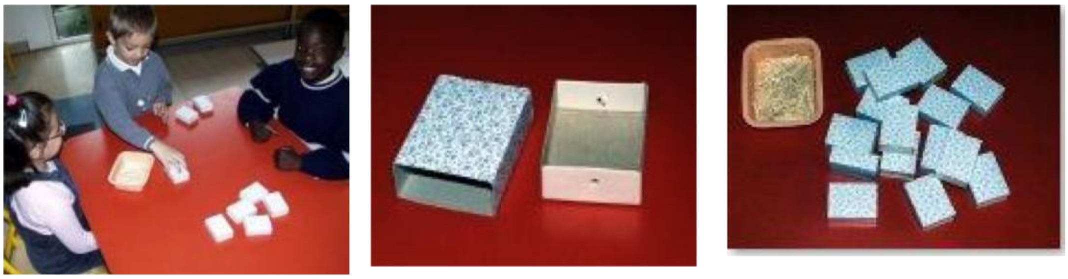

Figure 3 – The matchbox situation (pupils 4-5 years old)

A pupil has, just in front of her/him, several opaque matchboxes with a small hole (in order to put a match in the matchbox) and a lot of sticks (which are “safe matches”) in a plastic box. The task is the following: each pupil should put one and only one match in each matchbox and then declare that she/he has finished. When she/he thinks that she/he has successfully accomplished the task, another pupil (or two) comes to verify with her/him, by opening the matchboxes. The pupil has succeeded if there is one and only one match in each matchbox. It is a recurrent task that the teacher can organize for the pupils to work on their own, for instance from November to February.

Several didactical variables can be listed: the nature of the space (here, it is the table, a micro-espace); the number of boxes; the fact that the matchboxes can be moved or not.

DEVELOPPING REASONING AND LOGICAL SKILLS WITH THE DESIGNING AND READING OF LISTS

Several activities based on "lists" take part in the development of the logical thought in preschool: building a collection starting from a list, inventing the list as a tool in order to remember a collection or to communicate its content, building lists, building symbols in order to designate objects and then building a list etc. A list represents the easier way to designate non-structured collections of objects. Loubet & Salin (1999-2000) develop a set of situations (experimented with the COREM – Centre d'Observation et de Recherche sur l'Enseignement des Mathématiques, with several classes; all the experiments were analyzed with the tools of the Theory of Didactical Situations).

Introducing the idea of "list" for 3 year old pupils: a challenge

The following steps come from the research work of Loubet & Salin (1999-2000). It will be discussed during the working group.

1) To elaborate a collection: Doggy's suitcase

The main goal of the teacher here is to elaborate a common background for further situations in the classroom. The recurrent story is the following: Doggy is a puppet who brings a suitcase with various objects inside (3 the first day, up to 20 – it requires around 20 sessions and 2 months). The pupils look at the objects and they learn to name them. When all the pupils have touched and named the objects, the teacher puts these objects in the suitcase which is closed and Doggie explains that he will come the next day and bring a new object if and only if the pupils remember all the previous objects.

2) The game of the pictures: the pictures will now take the place of the objects, but the objects are still here. Each pupil gets a picture (at first 1, then 4, and finally 7) and has to take the matching object.
3) The game of the lists: the need to make up a list and to build the elements for its construction

This game takes two days and requires a collection of objects, and two boxes. The first day, the teacher takes some objects from the first box and puts them into the second box. Every pupil can look at these objects and touch them. The box is then closed, nobody can touch it. The second day, each pupil is asked by the teacher to play: if she/he names all the objects of the box, she/he has won. Otherwise she/he has failed. Each time the pupil names an object, the teacher shows her/him this object. If there are still objects in the box and if the pupil thinks that she/he has enumerated all the objects, the teacher shows her/him the objects too.

There are several didactical variables:

- the size of the hidden collections: when the number of hidden objects is small, the pupils can succeed using their memory. Memory is not useful when the number is big, hence the advantage of a list (to play with this didactical variable leads pupils to think that they have to find another idea to succeed);
- the construction of a list (invention of some pupils, diffusion in the classroom and role of the teacher): when pupils fail and think that their memory is inefficient, the role of the teacher is fundamental in order to encourage them to find another process. The knowledge now involved concerns the one-one link between the objects and their designation. It also depends on the following didactical variable;
- the composition of the collection (when the pupils have to construct the designation of the objects): a fundamental step here concerns the moment when pupils have to "decode" the written work of their friends.

CONCLUSION – CONSEQUENCES IN EDUCATION

Teaching requires knowledge that it does not support. Education is so far unable to make a didactic transposition of the enumeration leading to this knowledge. Therefore it is under the responsibility of the pupils. This generates difficulties for pupils and teachers. Current research (Margolinas, 2012 and in press) underscores that some kinds of knowledge such as enumeration are still "transparent", even if didactical works and official documents (Emprin & Emprin, 2010, pp. 27-29) point them out. And yet teachers have knowledge about enumeration and know about the difficulty involved by enumeration.

The role of the individualization of the processes of the pupils is also important (i.e. the teacher takes care of every process in her/his classroom). The devolution of the adidactical situations to the pupils is fundamental and requires a specific didactical contract. The validation of the processes of the pupils comes from the situation. What kind of posture the teacher should have? How can teachers design sequences integrating such (partially hidden) knowledge? The place and the role of oral and written formulations should be analyzed more precisely.

The role of the teacher is then twofold: he has to present the problem to the pupils, to make sure that its devolution is made, and to convince the pupils that they can succeed in performing the task and that the solution of the problem is up to them. Besides, the institutionalization can both concern knowledge (when it is identified, such as enumeration) and processes (lists). Moreover, recent research (Margolinas & Laparra, 2011) points out that some kinds of knowledge such as enumeration are also involved in other disciplines (French). Here is an example (Margolinas, in press): the pupils (5 years old) have to fill out the paper of Figure 4, using the tools of Figure 5 (scissors, a box, a pencil, and an alphabet).

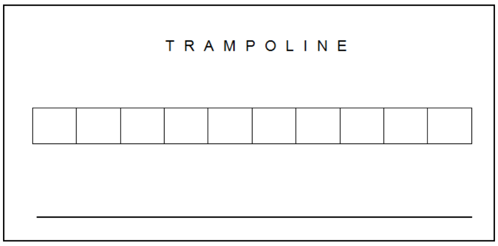

Figure 4

Figure 5

In order to stick the letters in the right position, pupils should make two enumerations (one with the first word "TRAMPOLINE" and one with the configuration of the individual letters-label). That underscores how enumeration is a transversal knowledge and not only a mathematical content. Further research should question more deeply this kind of knowledge and what it implies as far as pre-service and in-service teacher training are concerned. The case of the preschool and primary school with one teacher for several disciplines is a complex configuration, but it reveals transversal knowledge, when researchers coming from several disciplines work together.

NOTES

1. An *adidactical situation* is one in which the students is enabled to use some knowings to solve a problem "without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher]" (Brousseau, 1997, p. 30).

2. The notion of *devolution* is "the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility" (Brousseau, 1997, p. 230)

3. A *didactical variable* correspond to a potential (yet often implicit) choice for the teacher that modifies the accessibility of different strategies for solving the problem.

4. The teacher gives a cultural (mathematical) status to the new knowledge and he/she requests memorization of current conventions. He/she structures the definitions, theorems, proofs, pointing out what is fundamental and what is secondary.

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