TWO CHILDREN, THREE TASKS, ONE SET OF FIGURES: HIGHLIGHTING DIFFERENT ELEMENTS OF CHILDREN'S GEOMETRIC KNOWLEDGE[1]

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This paper presents three different geometrical tasks which involved the same set of geometrical figures. An analysis of the affordances and constraints of each task is discussed along with the results of two children's engagement with these tasks. Results indicated that not all children take advantage of the opportunities afforded by a given task and thus a combination of tasks is necessary in order to assess both strengths and weaknesses of children's geometric knowledge.

Key words: Geometry; tasks; preschool

INTRODUCTION

During the preschool years, children are developing and refining their spatial and geometric thinking (Clements, Swaminathan, Hannibal, & Sarama, 1999). Promoting geometric concepts and reasoning is also considered an important aim of several preschool programs (e.g. NCTM, 2006). Whether engaging in free play in a geometrically enriched environment or whether engaging in teacher-directed tasks, how can we know if and what children have learned from these activities? How can we know if we have achieved our goals? Ginsburg and Golbeck (2004) claimed that employing standardized procedures for measuring young children's learning is not appropriate. They note that many children are uncomfortable in or may be unfamiliar with the testing situation, and may display variable interest in the task. Instead, they suggested testing methods which might include clinical interviews and observations, methods that are "designed to be sensitive to the needs and peculiarities of young children" (p. 194). Yet, even when employing interviews and observations, children's competence may be linked to the specific nature of the tasks. What knowledge comes to the fore as children engage in different tasks?

In this article, we describe a study which aimed to investigate how different aspects of kindergarten children's geometric knowledge may become evident as they engage in different geometrical tasks. Using the context of two-dimensional figures, kindergarten children were presented with three different tasks. Two of the tasks employed the same set of geometrical figures, which was a subset of the figures employed in the third task. Keeping this in mind, we ask the following questions: (1) What elements of geometric knowledge are revealed by each task? Will the same elements come to fore in each task or will different elements be revealed in different tasks? (2) Will children display the same level of geometric reasoning (i.e. according to van Hiele) in different tasks or will different levels of reasoning be employed in

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different tasks? This paper presents in depth results of two children who took part in this study.

YOUNG CHILDREN'S GEOMETRICAL REASONING

Much of young children's knowledge, including their geometrical knowledge, is based on their perceptions of their surroundings. Later on, examples serve as a basis for both perceptible and non-perceptible attributes, ultimately leading to a concept based on its defining features. Such a process was described by Vinner and Hershkowitz (1980) who introduced the terms concept image and concept definition in reference to geometrical concepts. Visual representations, impressions, and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage. Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure may promote an immediate intuitive response not necessarily based on logical and deductive reasoning. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993, p. 14).

With regard to geometrical reasoning, van Hiele (1958) theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. At the most basic level, students' use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. Students at this level can name shapes and distinguish between similar looking shapes. Regarding naming, Markman (1989) proposed that when children hear a new name for an object, they assume it refers to the object in its entirety and not to its parts. In addition, children assume a given object will have one and only one name.

At the second van Hiele level students begin to notice the different attributes of different shapes but the attributes are not perceived as being related. Attributes may be critical or non-critical (Hershkowitz, 1989). In mathematics, critical attributes stem from the concept definition. Definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other properties may be reasoned out from the definition. Burger and Shaughnessy (1986) claimed that an individual's reference to non-critical attributes has an element of visual reasoning. Thus, they further claimed that a child using this reasoning may either be at van Hiele level one or at van Hiele level two, as he is pointing to a specific attribute, and not judging the figure as a whole.

At the third van Hiele level, relationships between attributes are perceived and definitions are meaningful. If the student points out that a figure is a quadrilateral because it has four sides and therefore it also has four angles and vertices, then that child may be operating at the third van Hiele level. Finally, we note that research has suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks (Burger & Shaughnessy, 1986).

METHODOLOGY

Participants

The two children reported on in this paper were both scheduled to enter first grade during the following school year. They learned in two different classes, but both classes participated in our program *Starting Right: Mathematics in Preschools*. This program integrated professional development for teachers with onsite guidance by a program staff member (Tsamir, Tirosh, Tabach, & Levenson, 2010). Throughout the year, teachers engaged students with various geometric activities to promote their knowledge of triangles, non-rectangular quadrilaterals, pentagons, hexagons, and circles. At the time of the study, children were expected to be familiar with the names of these shapes as well as with the mathematical language used to describe these shapes (e.g., vertices, straight and curved lines, open and closed shapes).

The set of figures

The set of two-dimensional figures which served as the context for this investigation included intuitive and non-intuitive examples and nonexamples of each of the shapes mentioned above. Figure 1 illustrates with triangles how figures may be grouped along two dimensions: a mathematical dimension and a psycho-didactical dimension.

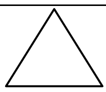
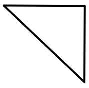
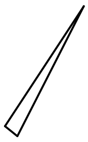
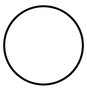
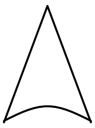
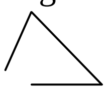
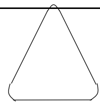
<table>
<tr><td>Dimensions</td><td colspan="5">Psycho-didactical</td></tr>
<tr><td>Mathematical</td><td>Intuitive</td><td colspan="4">Non-intuitive</td></tr>
<tr><td>Examples</td><td>
Prototypical triangle</td><td colspan="2"> Right triangle</td><td colspan="2"> Narrow scalene triangle</td></tr>
<tr><td>Non-examples</td><td>Circle
</td><td>“Triangle” with curved side
</td><td>Elongated pentagon
</td><td>Open "triangle"
</td><td>
Rounded "triangle"</td></tr>
</table>

Figure 1: Intuitive and non-intuitive triangles and non-triangles

Regarding the examples, the equilateral triangle, and possibly also the isosceles triangle, are considered prototypical of all triangles (Hershkowitz, 1989). Shaughnessy and Burger (1985) showed that triangles which are rotated, triangles without one side horizontal to the page, and very narrow scalene triangles are often not identified as triangles. These may be considered non-intuitive examples.

Regarding nonexamples, because the circle is intuitively recognized as such by even young children (Clements, Swaminathan, Hannibal, & Sarama, 1999) it may be considered an intuitive nonexample for a triangle. This reasoning holds true for other geometrical figures which the child can identify and name. Other easily identifiable nonexamples are those which are visually far removed from the prototypical triangle. On the other hand, nonexamples which are visually similar to the prototypical triangle may be considered non-intuitive nonexamples of that shape (Tsamir, Tirosh, & Levenson, 2008). Among this group of nonexamples, we specifically chose figures such that each nonexample would violate a different critical attribute. This allowed us to investigate the child's knowledge of each critical attribute separately. For example, in Figure 1, the rounded "triangle" is missing vertices, the "clown hat" has a curved side, the open "triangle" is not closed, and the stretched pentagon has five, instead of three, sides and vertices.

Examples and nonexamples for each of the other shapes were chosen in a similar manner, taking into consideration the necessity to limit the amount of figures presented at once to the children. The entire set of figures is presented in Figure 2 in the exact manner in which they were presented to the children for two of the tasks. The regular octagon, a shape that was not specifically taught to the children previously, was chosen as a non-intuitive nonexample for a hexagon, in order to investigate if the children would discern between shapes that had many sides, if they would actually count the sides, and not just group them together indiscriminately. It was also thought to possibly be a non-intuitive nonexample for a circle, especially the smaller octagon which was visually similar to a circle. In addition, we included a concave quadrilateral, pentagon, and hexagon as non-intuitive examples of each respective shape. The concave quadrilateral was drawn visually similar to a triangle and thus was considered to be a non-intuitive nonexample for a triangle.

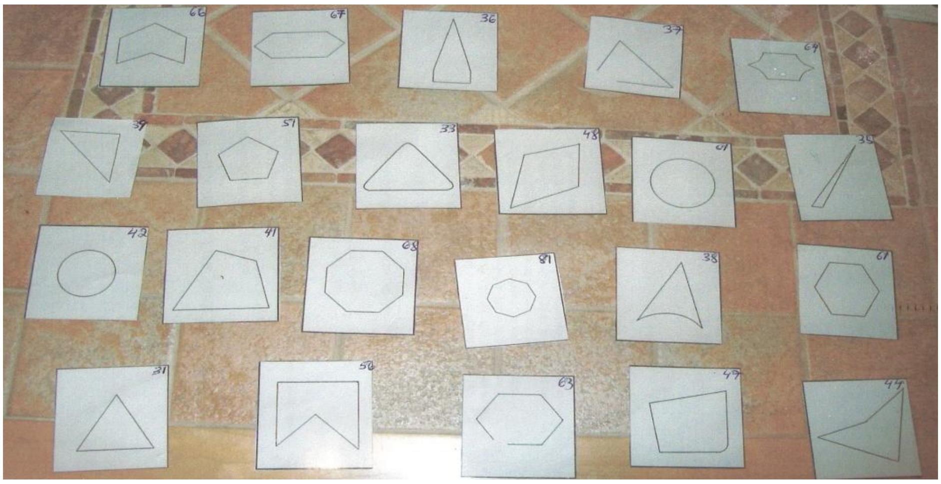


Figure 2: The entire set of figures.

The tasks

The tasks were presented to the children in the order presented below. Task two was implemented immediately following task one on the same day. Due to the age of the children, their ability to sit and concentrate, and other time constraints, the last activity was implemented a week later.

Task one: Free-sort. All 22 cards were placed on the table in front of the child as in Figure 2. The interviewer said, "There are lots of shapes on the table. I would like you to sort them. However you like. You decide which cards go together. You can put as many cards as you want together in the same group." As the child began working on the task, moving the cards around and grouping different cards together, the interviewer asked, "Why did you put those cards together?" The interviewer also reminded the child that he or she could make changes along the way, take away a card from one group and place it with another group and that a group may contain just one card if needed. When the child seemed to be finished, the interviewer asked, "Are you satisfied? Would you like to change anything?" When the child indicated that the sorting was done, the task was considered completed. This task was completely open-ended and had not been implemented with the children in their kindergarten class.

Task two: All-at-once. All 22 cards were placed on the table in front of the child as in Figure 2. The interviewer asked, "Is there a triangle here?" If the child answered yes, then the interviewer asked the child to point to the appropriate card without moving it. The interviewer then asked, "Is there another triangle here?" And again, the child was asked to point to it. This continued until the child indicated that there were no more triangles. The interviewer then asked the same set of questions with the same procedure for the quadrilateral, pentagon, hexagon, and circle in that order. This task was a completely closed task that was somewhat familiar to the children in the sense that they had practice identifying figures but had no experience dealing with 22 figures at once that could not be manipulated.

Task three: One-shape-at-a-time. This task differed from the previous two tasks in that the child considered one shape at a time. The interview began by considering only triangles. The interviewer held in her hand the cards with the figures shown above in Figure 1. The cards were presented one at a time and each child was asked: Is this a triangle? Why? The child was allowed to take hold of the card, rotate it, and take his time to consider the one figure. After the child answered, the interviewer took back the card and placed another on the table. The same questions were repeated for each card.

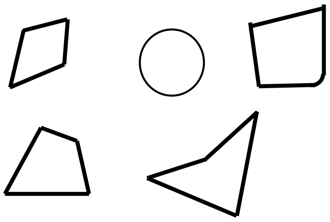
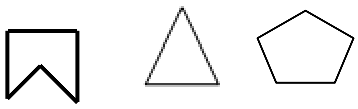
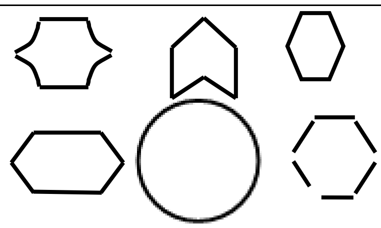
Is this a quadrilateral?	Is this a pentagon?	Is this a hexagon?
		

Figure 3: Is this a…?

When this set of cards was completed, the interviewer went on to quadrilaterals, pentagons, and hexagons using the figures in Figure 3. The same set of questions was repeated each time. This task was familiar to the children in that they had much practice in identifying one figure at a time and explaining their reasoning.

RESULTS

Johnny

On the *Free-sort task*, Johnny built the groups shown in Figure 4. In his words, there was a group of circles, triangles, quadrilaterals, pentagons, hexagons, not triangles (there are two groups of not triangles), not quadrilaterals, not pentagons, not hexagons, and one group for which he has no name. In general, it seems that Johnny sorts the figures according to geometrical shapes naming the figures as he goes along. At no time during this activity did he mention explicitly any critical or non-critical attributes of the figures presented.

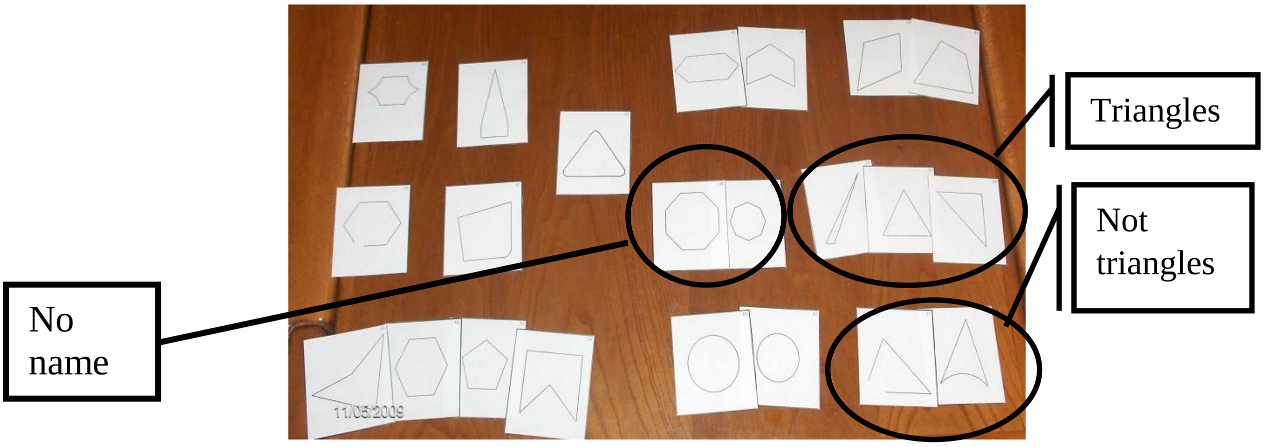


Figure 4: Johnny's sorting of the 22 figures

Within the group of "pentagons" we note that Johnny included two figures which were not pentagons, the regular hexagon and the concave quadrilateral. Yet, Johnny does not include any open figures or figures with curved lines in his groupings of the different polygons. In other words, he does not include non-polygon figures with polygons.

The first group of cards which does not consist of what Johnny terms as examples of geometric shapes includes the triangle-like shape that is not closed and the triangle-like figure with a curved side. Johnny groups together these shapes and says "not triangles". What does he mean here? There are many other figures which are also not triangles. The group of circles and the group of hexagons are also not triangles. In addition, the "not-triangle" figures are also not pentagons and not hexagons. Yet, he chooses to relate to them as "not triangles". What's more, he later relates to the quadrilateral-like figure with a rounded corner, as a "not quadrilateral". Perhaps Johnny is trying to tell us that these figures look like triangles or quadrilaterals but that he knows that they are not triangles and quadrilaterals. Perhaps Johnny is naming these figures "not triangles" and "not quadrilaterals". If this is the case, we may surmise that the name of the figure is the criterion Johnny used in his sorting and that Johnny has consistently used one criterion throughout this sorting. It is also possible that Johnny is sorting the figures according to geometrical shapes and then subdividing them into examples and nonexamples. In other words, open and curved figures are identified in relation to the polygon they most closely resemble.

On the *All-at-once task*, For the most part, Johnny identified correctly all of the shapes. There were three exceptions. When asked to identify pentagons, he failed to point to the regular pentagon and to the concave pentagon. In addition, he pointed twice to the concave hexagon – as a pentagon and as a hexagon.

On the *One-shape-at-a-time task*, Johnny correctly identified all of the figures on each sub-task. For each figure presented to Johnny, he consistently noted both the number of vertices and the number of sides. When the figure was open or had curved lines, he correctly identified the figure as a nonexample of the requested shape, explicitly referring to the critical attribute which was violated. From this task it seems that Johnny has a wide concept image of the figures presented.

To summarize, Johnny's ability to name the figures became known only from the *Free-sort* task. This is important because some children are able to identify a shape when given the name but may not be able to name the shape on their own. From the same task, we get a sense that for Johnny, nonexamples are connected to examples. In other words, a nonexample is not generic but related to some specific figure which it is not an example of. This too, is information about Johnny that we did not learn from the other tasks. Both from the *Free-sort* task and the *All-at-once* task, it seems that Johnny's knowledge of pentagons is less stable than that of other figures. Yet, on the pentagon sub-task of the *One-shape-at-a-time* task, he made no errors. Perhaps handling 22 figures at once raised the level of difficulty for Johnny, or perhaps, he merely has a narrow concept image of pentagons. Finally, if we had only engaged Johnny in the *Free-sort* task, we might have surmised that he reasons visually with geometrical figures, actions indicative of reasoning at the first van Hiele level. Only from the *One-shape-at-a-time* task were we able to learn that Johnny is capable of using mathematical language and critical attribute reasoning, indicative of the second level of geometrical reasoning according to van Hiele.

Randy

Randy's final groupings on the *Free-sort* task is presented in Figure 5. In her words, there is a group of figures with eight vertices, six vertices, five vertices, four vertices, three vertices, two vertices, and a group without any vertices. Throughout, the only shape Randy explicitly names is the circle. She placed the rounded corner triangles with the circles because "it too does not have any vertices." She does not say that this shape is a circle nor does she say, for example, that the open "hexagon" is a pentagon because it has five vertices. Rather, she consistently counts points, or what she calls vertices, and groups together figures according to their number. Her concept image of a vertex is somewhat blurry. She claims that both the regular hexagon and the curved "hexagon" have six vertices. However, the point connecting curved lines is not called a vertex. When discussing the "triangle" with a curved side, she points to the endpoints of the curved side and asks the interviewer, “Are these vertices?" When the interviewer does not respond, Randy goes on, “I think not. Because this is a curved line. But, let's say that there are three [vertices]. In other words, Randy notes that the figure has a curved line and questions the legitimacy of calling the points, vertices. Yet, she decides to act as if they are vertices and continues sorting along this line of reasoning. In the end, Randy sorted all the figures by one critical attribute – the number of vertices. (The only time she mentions the sides is when grouping the triangles.) In doing so, the world of geometrical figures, with its examples and nonexamples, becomes somewhat blurred.

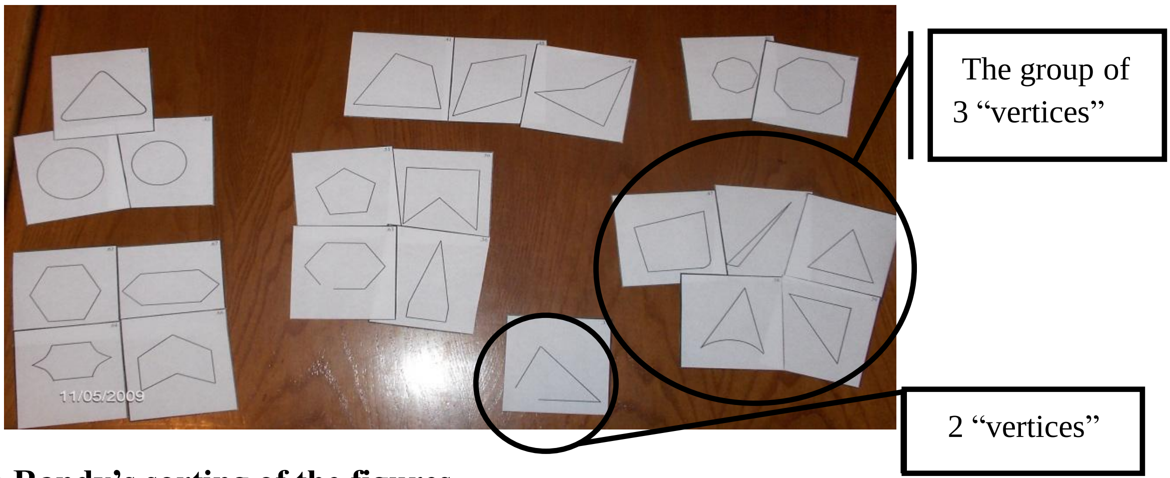


Figure 5: Randy’s sorting of the figures

On the *All-at-once task*, Randy correctly identified all figures except for the concave quadrilateral, which she missed pointing to. On the *One shape at a time* task, she correctly identified all of the figures. She not only related to sides and vertices but also to the figures being closed. For example, when identifying the prototypical triangle, Randy comments, "It has 3 vertices, 3 sides, and it is closed." Reference to closure only took place on the triangle sub-task. When reasoning about quadrilaterals, Randy referred only to the sides and vertices while for the pentagons and hexagons she referred only to the vertices. When presented with a nonexample, she explicitly referred to the critical attribute which was violated using correct mathematical language to express her reasoning. Interestingly, when identifying the open "hexagon", Randy claimed "it is open and it has 5 vertices." In other words, although the figure is open, Randy still counted the points. For the curved “hexagon” Randy said that it is not a hexagon “because it has six vertices but only two sides."

To summarize, from all three tasks, we see that vertices play a dominant role in Randy’s geometric reasoning. This was first evident from the *Free-sort* task and was backed up by the *One-shape-at-a-time* task, where slowly but surely, vertices were the only remaining critical attribute mentioned. If we had only implemented the sorting activity with Randy, we may not have learned that she is aware of other critical attributes. We may also not have learned that Randy is capable of identifying figures or we may have thought that handling 22 figures at once was too much. Yet, on the *All-at-once* task, she identified all but one figure. On the other hand, if we had only implemented the *One-shape-at-time* task, we may not have learned of Randy's dilemma regarding points versus vertices. We may also have missed that for Randy the number of vertices is perhaps more critical than the other critical attributes.

DISCUSSION

As mentioned in the background section, assessment of children's geometric knowledge may vary greatly depending on the set of examples and nonexamples used in the study. However, in this study, the set of figures was constant. What varied, were the tasks themselves. Thus, as we look back on the results of this study, we focus on the tasks themselves, their similarities and differences, their affordances and constraints, and the knowledge which each task brought to light.

The *Free-sort* task was both open-ended and unfamiliar. As an open-ended task, there was no correct or incorrect way for the children to sort the figures. Yet, different aspects of children's geometric knowledge could still be assessed. Ability to name figures was one such aspect. This task also allowed us to investigate the relationships children might perceive between figures and the generalizations children might have constructed along the way. This is in accordance with Lane (1993) who claimed that tasks of this nature may shed light on the cognitive processes that underlie performance, such as discerning mathematical relations, organizing information, evaluating the reasonableness of answers, generalizing results, and justifying an answer or procedure. Johnny's groupings of "not triangles" and "not pentagons" may reflect that Johnny relates nonexamples to examples. That is, a figure is not merely a nonexample. It is a nonexample related to some specific shape.

The *All-at-once* task was a closed task. Children could either correctly or incorrectly identify each figure. The obvious constraint of this task was the set of figures. While children were familiar with the task of identifying various figures they had no experience dealing with 22 figures at once that could not be manipulated. This was a new challenge. Thus, the task afforded us a glimpse into which figures may be confused with other figures (such as the confusion between pentagons and hexagons evident by Johnny). This task was also the only task where the child's response was indirectly challenged. Recall that even after the child had pointed to all of the triangles (or quadrilaterals or pentagons, etc.) he was asked yet again if he could identify another triangle. Thus, this task also afforded us a glimpse into the child's confidence in his ability to identify figures.

The *One-at-a-time* task was both closed and open in that it began with an identification which was either correct or incorrect but continued with an explanation which was more open in nature. The closed nature of both this task and the *All-at-once* task and the focus on identifying figures on both tasks allowed us to assess which figures children may find more difficult to identify. Having children explain their reasoning on the *One-at-a-time* task afforded us the opportunity to gain insight into the van Hiele levels at which they might be operating. Explanations also brought to light children’s knowledge of appropriate mathematical language.

As noted above, different tasks afford children different opportunities to use and display geometric knowledge. But, not all children take advantage of the opportunities afforded by a given task. If a child only employs visual reasoning on one task, can we conclude that he or she is operating only at the first van Hiele level of reasoning? Previous research suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks (Burger & Shaughnessy, 1986). This study supports this suggestion as knowledge and reasoning which did not necessarily come to the fore on one task, sometimes appeared on another task. Thus, we conclude, a combination of tasks is advantageous when assessing both strengths and weaknesses of children's geometric knowledge. Our challenge as mathematics educators and researchers is to continue analysing the affordances and constraints of different tasks in order to optimize learning experiences.

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