GAME PROMOTING EARLY GENERALIZATION[1]

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The paper presents and studies a case of generalization appeared during an activity based on a game, proposed to pupils 5-6 years old. The analysis of pupil's behaviours in front of the task furnishes some examples that prove the possibility of an early and significant mathematical activity of generalization.

Keywords: play, children's experiences, relational thinking, preschool.

THEORETICAL FRAMEWORK

In early years mathematics the relations between play and learning are object of investigation and study. In particular, Bartolini Bussi (2008, p. 262-263) emphasises that:

> "The collective play is important both as socializing element, and as "instrument of production" of problematic situations. Fundamental is the encounter with the play with rules; it requires the overtaking and the solution of problems that we can relate to three areas."

These three areas are: language (understanding, presentation and invention of new rules), socialization (respect of rules) and abilities of mathematical kind (order and behaviour organization).

About the relations between play and learning, Schuler (2011) observes that theoretical models and empirical researches confirm the evidence of learning while playing. But learning is not without requirements and depends on situational conditions. She poses some interesting questions:

> "Consequently the underlying question seems not to be "Can children learn while playing?" but rather "How can learning while playing be modeled?" and "Can children learn mathematics while playing?" (2011, p. 1913).

In particular, she studies situational conditions of learning while playing and she highlights three main blocks: affordance, liability and conversational management:

> "[…] rules can offer mathematical activities beyond a material's intuitive affordance and thus create liability. Intuitive affordance of materials is replaced in games by (*the affordance of*) keeping the rules and winning *the game*" (Schuler, 2011, p. 1919).

Moreover she emphases "the central role of the educator". In fact, sometimes it is difficult to adopt a good equilibrium between a free and spontaneous play and a

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guided play. In other words, “Play is not enough. […] children need adult guidance to reach their full potential” (Balfanz et al., 2003), but when the teacher proposes a play finalised to promote particular abilities, he risks to force in some way the child and to impose directions of work connected with the play finality.

What is the meaning of the words ‘generalization’ and ‘abstraction’ in mathematics? Generalization is cited as typical form of mathematical thinking. Very often it is related to algebraic reasoning, but it is possible to observe the use of generalizations also in other mathematical activities. Generalization is often associated with abstraction, but, Mac Lane (1986) writes:

> “Generalization and abstraction, though closely related, are best distinguished. A “generalization” is intended to subsume all the prior instances under some common view which includes the major properties of all those instances. An “abstraction” is intended to pick out certain central aspects of the prior instances, and to free them from aspects extraneous to the purpose at hand. Thus abstraction is likely to lead to the description and analysis of new and more “abstract” mathematical concepts.” (p. 435-436).

Malara (2012, p. 57-58) describes the ‘generalization process’ as:

> “… a sequence of acts of thinking which lead a subject to *recognize*, by analyzing individual cases, the occurrence of common peculiar elements; to *shift attention* from individual cases to the totally of possible cases and *extend* to that totality the common features previously identified”.

Also Mac Lane (1986) speaks about “shift of attention”, but referring to abstraction:

> “*Abstraction by shift of attention*. Some abstractions arise when the study of a Mathematical situation gradually makes it clear that certain features of the situation – perhaps features which were originally obscure – are really the main carriers of the structure. These features, with their properties, are then suitably abstracted” (p. 437).

In his theory of ‘universal model’ Hejny (2004) distinguishes six different stages: motivation, isolated (mental) models, generalisation, universal (mental) model(s), abstraction, abstract knowledge. In particular, concerning the ‘Stage of generalisation’ he writes:

> “The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalising all these models is discovered. The stage of generalisation does not change the level of the abstraction of thinking” (Hejny, 2004, p.2).

Moreover Hejny (2004), describes the ‘Stage of abstraction’, as a stage in which a new concept, process or scheme is constructed, bringing a new piece of knowledge.

Dörfler (1991) differentiates two forms of generalization: empirical and theoretical.

> “The basic process in **empirical generalization** is to find a common quality or property among several or many objects or situations and to notice and record these qualities as being common and general to these objects or situations. The common quality is found by comparing the objects or situations, with regards to their outward appearance, isolated mentally, and detached from the objects and situations.
>
> In contradistinction to this form Dörfler introduces another one - called **theoretical generalization** – and describes it with the help of a theoretical model for processes of abstraction and generalization which can often lead to the genuinely mathematical concepts (propositions, proofs, etc.)” (Ciosek, 2012, p. 38).

Dörfler criticizes the first, since it realizes only a recognition process, while the second can contribute to the concepts construction. In its “model of the processes of abstraction and generalization” he emphasises the role of “invariants of actions” that emerge as a consequence of the repetition of actions and define a “schema”.

In the present paper we chose to report some definitions of ‘generalization’ and ‘abstraction’ since there are concepts very difficult and their meanings depend from the theories. In the following part of the paper we will refer to these quotations, with the aim to explain our ideas and results.

RESEARCH QUESTION

The present research is placed in the theoretical framework of early mathematical education by play. In particular, it deals with development of reasoning promoted in children from a game (play with rules). The initial hypothesis is that a suitable play can produce an early and spontaneous use of generalization. Our aim is to give answers to the following question:

> “Is it possible to promote in children the recourse to generalization in a play context?”.

METHODOLOGY

The participants

The experiment took place in the last year of kindergarten (pupils 5-6 years old), in school years 2010/2011 and 2011/12, working on the topic around once for week. It involved respectively 15 and 13 children. They worked sometimes in groups (7-8 pupils), under the guide of the teacher[2], sometimes individually. Whole activities took place in presence of a researcher (the author of the present paper).

The game of coloured houses

We present only a part of a wider research based on a game (without winner), named from the pupils “The play of coloured houses’, invented from the author of the

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2. I wish to thank the teacher Palma Rosa Micheli (Scuola dell'Infanzia Statale "Lodesana", Fidenza (PR), Italy), for her collaboration and helpfulness.

present paper[3] (Vighi, 2010). The game is based on a disposition of houses with three different colours (red, yellow, green) in a grid 3x3, respecting the following rule: in each row and in each column it needs to have houses of three different colours. The possible different villages constructed following this rule are twelve. We report here some examples:

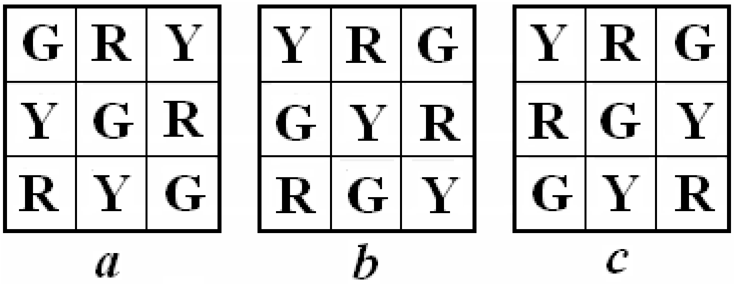


Figure 1: Examples of villages

The game can be seen as a simplified version of Sudoku, with a grid 3x3 (instead of 9x9) and only three ‘symbols’ (it is possible to use also digits 1, 2, 3 or letters in place of colours). From the mathematical point of view, it is a ‘Latin Square’, an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

The game requires the contemporaneous management of rows, columns and colours. Rożek & Urbanska (1998) studied in depth row-column arrangements and they emphasized the related difficulties. In particular, they write:

> “Children have a different awareness of the rows and columns arrangement. Some of them prefer rows, some of them columns. It appears that it was difficult to see both rows and columns, especially for young children” (p. 304).

The game can be executed by means of ‘method of trials and errors’ or using rules discovered during the game: “It is impossible to have a red house here”, or “Here it must be a yellow house” etc. When a pupil plays, he makes argumentations, and he uses hypothetic-deductive reasoning: “If I put here a green house, then …” and so on.

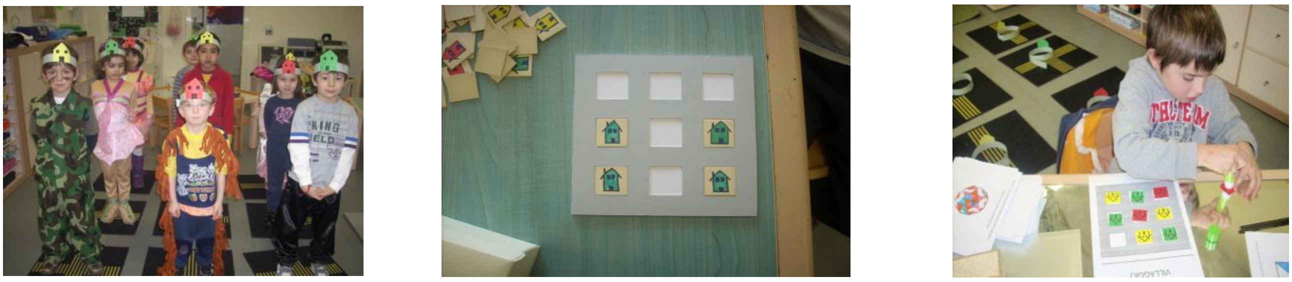
The procedure

The activities took place in every day context. Teacher presented the game and she conducted it, promoting and fostering the viewpoints of children, without force their thinking, but waiting to listen their ideas. Researcher observed, recording on video, later she analyzed and transcribed dialogues, making also written observations.

In a first time teacher worked on the linguistic level: to understand the rules of the game it needs to clarify the meaning of the words “row” and “column” and their use. Afterwards she presented the game in a context of motor activity (Fig. 2a), with the aims to involve pupils, to promote understanding and application of rules and

[3] The first idea of this game bore during a conversation with prof. E. Swoboda (University of Rzeszów, Poland).

socialisation also. In a second time pupils played with coloured tiles and appropriate supports: small cardboards to fit in appropriate grooves (Fig. 2b) or to glue in a sheet of paper opportunely arranged (Fig.2c). At last teacher proposed a conversation and a discussion about the constructed villages and their features.



a b c

Figure 2: different ways to play with coloured houses

ANALYSIS OF CHILDREN BEHAVIOURS

We present here only a brief qualitative analysis of the activity concerning village's construction with tiles.

A first important observation is about different behaviours, emerged from the analysis of videotapes: in some cases the child brings a tile and after he puts it in a box, in others cases the 'anticipatory thinking' leads to observe before the map of the village and after to choose an useful tile.

The village constructions were very different. Some pupils worked realizing systematically rows, starting from the top and proceeding without hesitation from the left to the right, constructing before the first row and afterwards to the second and the third; in this way, the game became a simple exercise. But, as we forecast, most children had difficulties to coordinate rows and columns: some respected rules only in rows, others only in columns, others in part in row and in part in columns, jumping from the one to the other. Some pupils putted tiles on the grid without any organization. Sometimes rules were respected only locally, so, when the village was finished, its global construction was wrong. Sometimes the child arrived in front of an impossible situation, for instance he cannot finish your village since no colour can be used without make mistake. It is an interesting case: it presents the concept of 'impossible', important in mathematics, it forces to cogitate and to formulate hypothesis of change, promoting hypothetical-deductive reasoning. For example, we report here the sequence of construction followed by Mattia:

Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G
					Y	R		Y	R		Y	R		Y	R		Y
										G			G	R	Y	G	R

Figure 3: Mattia's construction

At this point Mattia stops and, observing the first column, he says: "There are two yellow". So, he uses the number to explain why the column is incorrect. He decides to change some tiles until his village is right.

The final activity of control was often suggested from the teacher. In many cases we observe that if the first construction of the village is wrong, for children it is difficult to come back and to modify their own previous execution. Child, invited to discover how to control her/his own village and eventually to modify it, showed different reactions: refusal or inability to continue, procedures based on trials and errors, fast and opportune individuation of modifications. In fact, as Vygotskij and Luria (1987) observed, to show to the pupils the contradictions present in their conclusions creates a big difficulty in pupils, but obviously it depends on didactical contract. On the contrary, as we wrote before, if a child works placing systematically the tiles in rows or in columns, the game becomes easy and no problematic.

THE 'LITTLE LADDER THEOREM'

After the village's realization, teacher decided to put the sheets of paper with villages (Fig 2c) on the walls of the classroom and she suggested an activity of comparison of pupil's productions. It is well known that the activity of comparison is fundamental in mathematics, to construct concepts: to think about analogies and differences can promote it. During the activity of comparison a lot of suggestions emerged: a local way of seeing leads to observe only some couples of tiles with the same colours, placed in the same places ("In the first village there is a green house here, in the second also"), while a global way to see promotes the individuation of "equal villages" or "symmetric villages" etc.

In particular, as Brousseau (1983) highlighted, the passage from the work on the desk (micro-space) to the observation of the classroom walls (meso-space), along with the visual perception stimulated by colours, promoted an important 'shift of attention' (Malara, 2012) from rows and columns, explicitly mentioned from the rules of the play, to the monochromatic diagonal present in each village.

It is an example of generalization in the meaning of "empirical generalization" (Dörfler, 1991), but also in which the structure appears as generalizing isolated models in sense of Hejny (2004): the creation of connections between isolated models, referring to a particular disposition of tiles independently from their colour, produced generalization. Some pupils said that "Villages have three houses disposed as a ladder, with the same colour", others observed villages and confirmed this conjecture. To indicate the monochromatic diagonal discovered in each village some pupils used the locution "bandy row", others 'arrow', some others "little ladder" since the disposition of tiles suggested the steps of a small ladder and so on.

So, in this way, pupils discovered and formulated a theorem: "In each village, there is a diagonal with houses of the same colour". We named it the 'Theorem of little ladder'. Afterwards some of them observed the presence of two different kinds of diagonals: "from down to up or vice-versa". It promotes the observations of both the diagonals and the formulation of another conjecture: "In the other diagonal there are three different colours”, but this was perceived as no relevant aspect, since it seems to respect the play rule.

FROM GENERALIZATION TO ABSTRACTION

In a second time, we decide to investigate if children use or not ‘diagonal rule’ in the following activities. We observed two different kinds of behaviours. The first is well illustrated from Chiara’s strategy (Fig. 4).

Figure 4: Chiara’s strategy

Chiara starts with a yellow diagonal, so using the ‘little ladder theorem’, she continued with two correct passages, after she makes a mistake that leads to have at the end a ‘wrong village’. When the teacher asks to control if the village is correct or not, Chiara realizes that in the third row there are two green tiles near, but she don’t perceive the error present in the second row, since the red tiles not are next to each other. Chiara remember and use ‘small ladder theorem’, but her execution shows mistakes and a difficult management of play rules. In fact, for Chiara the presence of a monochromatic ladder is only a new rule of the play, a starting point for her activity.

In the school year 2011/2012, we decided to submit to pupils of preschool (5-6 years old) a new version of the game, in three dimensions: it consisted in the construction of a ‘palace’ of three floors (a cube 3x3x3), using 27 cubes, with similar rules: “In each wall face it needs to have three different colours in each row and in each column”. In this case, tiles were replaced from cubes of three different colours. Really cubes are more similar to houses and, as research documents, to work in 3D dimension is suitable for young pupils. The complete experience is described in (Vighi, 2012) (See Poster Presentation in CERME 8). The village construction was realized putting cubes (red or yellow or blue) one near to the other in a virtual grid three times three (Fig. 5).

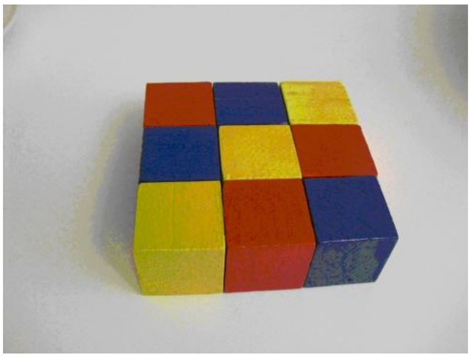


Figure 5: Village of coloured cubes

This choice promoted an important breakthrough, since, as Rożek (1997) writes, in a row-column arrangement of figures the distance between objects influences in depth the observation. After the village construction with cubes, it happened that a child observed the presence of a yellow line (Fig.5), suggested from the colour and also from the idea of "straight line" and he said that "there was a mistake". Teacher suggested that in fact all rows and columns respected rules and the child replies that "Yellow cubes are in arrow" ('little ladder' in the previous experience).

Another important breakthrough happened when a child observed that in the other diagonal there were three different colours: he indicates it with his hand, accompanying with gesture and sound: "here, blue, yellow, red, pum, pum, pum" and he added: "An 'arrow' entirely yellow, another of three colours, it is an 'eks'!". In other words, the disposition of diagonals suggested the mental image of letter X. Its presence in all villages produced a passage from isolated models to a general model in the meaning of Hejny (2004), but … not only this.

Of consequence of "X discovery", some children changed their way of village construction, also in the following activities. They started putting cubes in an 'X disposition' and completing the remaining parts. Obviously, in this way the game becomes easier: the construction of a coloured village changes a lot, since starting from diagonals, the placement of the other houses is obliged. We describe here, for instance, Nicholas strategy (Fig. 6):

Y		
	Y	
		Y

Y		B
	Y	
R		Y

Y	R	B
B	Y	R
R	B	Y

Figure 6: Nicholas strategy

There is a fundament difference between the use of one diagonal or two diagonals as starting point in village construction: in the first case (Chiara example) it is only a use of a new rule obtaining by generalization, in the second (Nicholas example) it appears another important and very strong 'shift of attention', but related to abstraction, in the sense of Mac Lane (1986): 'letter X' is understood as one of the "main carriers of the structure". It is an example of "theoretical generalization" in the meaning of Dörfler (1991): the 'letter X rule' becomes an "invariant of action" and it creates a "schema" of action.

This finding of X was unexpected also for us. As we said before, when we planned the activity our interest was on row-column arrangements, early reasoning etc. But it was very surprising for us that, after its discovery, some pupils chose to adopt and use the method of 'letter X' in the following activities on this topic. In particular, afterwards some children (38%) always used the 'rule of two diagonals', 32% the 'rule of a diagonal', and the remaining 30% other various methodologies.

The main result of the second experiment, maybe promoted from the use of wooden cubes, is the use of generalization and abstraction promoted from the “X discovery”. In the previous experiment with tiles we haven’t this important shift of attention on the structure of villages, the rule of letter X remained unknown.

CONCLUSIONS

The “Play of coloured houses” is surely motivating for pupils: speaking with the teacher, I knew that they often proposed the game at home to their parents or brothers. It is a first goal; in Hejny’s theory the “motivation” is the first stage of development and structuring of knowledge.

The experimentation furnished positive answers to our research question. In particular, the mathematical learning is related to understand and to use mathematical methods. So, learning is realized on meta-mathematical level, since it is related to generalization processes, described from Malara (2012):

> “… the key element in these processes is … the *shift of attention* from individual cases to all the possible ones, as well as the extension and adaptation of the model to any of them” (p.58)

The recognition of an “invariant” or “schema” and to use of “letter X” to describe it, is a “symbolic description” in the Dörfler’s theory meaning. Moreover, the use of X-strategy, we think, means to pass from empirical to theoretical generalization: for these young pupils, as for adults, generalization (and its consequences) becomes an instrument of work.

When we plan this game and its implementation in preschool, our aims were far from to study early generalisation and abstraction, but generalization appeared with the discovery of a monochromatic diagonal in each village and the use of X-strategy, we think, promoted to pass from generalization to abstraction: for these young pupils, as for adults, generalization (and its consequences) becomes an instrument of work.

About methodology, we agree with Schuler (2011): “Potentially suitable materials and games need a competent educator with regard to didactical and conversational aspects”. In our experience the role of the teacher and a conversational management appeared determinant. We want to add another important conclusion about “conversational management”: as we wrote before, the game was entirely conducted from the teacher with the presence of researcher as observer. The latter had the possibility to “peek and catch” some observations made from children, while the teacher was involved in the action. After an exchange of opinions with the observer, teacher took advantage from these suggestions and she used them in the following activities and conversational managements, improving in this way the activity in classroom.

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