DEVELOPMENT OF STUDENTS' UNDERSTANDING OF THE THRESHOLD CONCEPT OF FUNCTION

Kerstin Pettersson, Erika Stadler and Torbjörn Tambour

Stockholm University

In this paper we report from a study of university students' understanding of the threshold concept of function from a discursive perspective. The research question was what kind of changes of the interlocutors' mathematical discourse that could be observed during a study year. Data has been collected through observations, questionnaires and interviews during the students' first year of mathematics courses. As an analytical tool we used the commognitive framework of Sfard. The results show that the students expanded their use of mathematical words. Visual mediators were also important for students' understandings, but it was not enough to stay within an understanding strongly leaning on the visual.

THE THRESHOLD CONCEPT OF FUNCTION

For students who have recently entered university studies in mathematics the development of understanding of mathematical concepts seem to be a cucial issue (Thomas et al., 2012). Over the years, the meaning of mathematical understanding has shifted from focusing individual cognitive aspects to socio-cultural and situated perspective where the learning context and discursive elements are focused. Being aware of that the students do learn mathematics in a social setting, we have chosen to bring back focus on the relation between individual students and their understanding of mathematics. More specifically, we are interested in novice university students' understanding of the mathematical concept of function.

Limited understanding of the function concept has been shown to have an effect on the transition from secondary level to university (Thomas et al., 2012). Moreover, functions are used in all branches of mathematics and it is a concept that students encounter early during their school time. The first encounter with functions may be on a very informal and intuitive way but rather soon the students may work with functions as algebraic expressions, such as $y = 2x – 1$, and graphical representations. In university mathematics, students encounter a more formal definition that distinguishes functions from other relations is that y in $y = f(x)$ is uniquely determined by x. The increasing abstraction and generality of functions may be one reason for that students' understanding of the concept of function poses great difficulties for students' learning (e.g. Artigue, Batanero, & Kent, 2007). Even if functions are almost invariably introduced as a process, students also need to use functions as objects (Sfard, 1991). This reification of the process into an object is known to be hard for the students (e.g. Viirman, Attorps, & Tossavainen, 2010).

The notion *threshold concept* was introduced by Meyer and Land (2005) for initially troublesome concepts for which students' development of understanding of the concept involves a potential to transform the understanding not only of the concept, but also the understanding of the subject area where the concept is included. A threshold concept can be seen as a portal to a new and previously unreachable understanding. Meyer and Land characterized threshold concepts as initially troublesome, transformative, integrative and irreversible. To come to understand a threshold concept is troublesome for the students, it is a threshold to cross over, but then their understanding gets transformed. Understanding a threshold concept will bring a significant shift in students' perception of the subject. The transformation may be sudden but it often occurs over a long period. Integrating prior understandings is part of the transformation and the understanding is irreversible; when it has been transformed to a new one, it will not be forgotten or unlearned without a big effort. There are several threshold concepts in mathematics (Pettersson, 2011). However, studies of mathematical threshold concepts are rare (Pettersson, 2011) and there is also a need for more general discussions about the transformation (Scheja & Pettersson, 2010). In this paper we focus the threshold concept of function and the transformation of students' understanding. There have been some discussions about how to conceptualize the transformation when crossing the threshold (e.g. Scheja & Pettersson, 2010). This study aims to further explore this process of transformation.

THE COMMOGNITIVE PERSPECTIVE

The notion of threshold concepts provides a theoretical perspective on the character of students' understanding of specific mathematical concepts that play a particularly important role for students' access to university mathematics (Pettersson, 2011). However, to study students' development of their understanding of threshold concepts, we also need a theoretical framework that focuses students' learning of mathematics from a more general point of view. Threshold concepts are often presented by mathematicians or in mathematical literature with a significantly different mathematical discourse than that of the students. Thus, to study students' development of their learning and understanding in terms of discourses seems relevant. Therefore, we have chosen to use Sfard's commognitive perspective to analyse the students' gradual concept development of functions. In our analysis, we will use the interrelated features that according to Sfard make a mathematical discourse distinct from other kinds of discourses. These are mathematical words, visual mediators, narratives and routines (Sfard, 2007).

A mathematical discourse implies the use of *mathematical words*. There are many words that students almost only meet in a mathematical classroom context, but there are also colloquial words that get another meaning when used in a mathematical discourse. The concept of function is in itself a mathematical word and the very core of the notion of threshold concepts is that the meaning of the mathematical word function dramatically changes when a student crosses the threshold. *Visual mediators*, such as symbolic artefacts and manipulatives, can be used as a cursor on the objects of communication. Graphs are used as visual mediators for functions and can be used to show features and behaviours of different functions. The crossing of a threshold may be recognized by identifying new ways of using visual mediators that are related to functions. *Narratives* are descriptions or accounts of objects. It is any written or spoken text that is used within the discourse and can be subject to endorsement, i.e. narratives can be judged as true or false. In the interviews, the students make use of narratives to describe their understanding of the function concept. Thus, narratives are tightly related to the development of mathematical understanding and the learning of specific mathematical concepts. *Routines* refer to repetitive patterns for actions and serve as important tools for the interlocutors to participate in any kind of mathematical discourse. Routines can be due to properties of mathematical objects. Concerning functions, a familiar routine from secondary level is to examine functions with tables of values and graphs and to discuss the behaviour of a function.

In our study, we will use the notions of mathematical words, visual mediators, narratives and routines to analyse university students' gradual development of their mathematical discourse about mathematical functions. What kind of changes of the interlocutors' mathematical discourse could be observed during the transformation of the understanding of the threshold concept of function?

METHOD

To capture the process of transformation of students' understandings of a threshold concept, a process that mostly is a development over a long period (Meyer & Land, 2005), the students have been followed during a study year. The students were student teachers at a Swedish university taking courses in mathematics. In total the study included 18 students, all the students that followed the mathematics courses for prospective mathematics teachers in their second semester. These students also included a few students taking the courses a second time and one in-service teacher expanding her mathematical competence. In this paper four students, A, B, C and D, are followed. These four are chosen since they have taken part in nearly all of the data collection occasions (only interview 3 is missing for C and interview 4 for A) and in that way have given a rich dataset. B and D are first year student teachers, A is an in-service teacher and C is taking the courses a second time. The four students followed the same courses with the same lecturers.

During the first semester of their studies the students took courses in general education and an introductory course in mathematics. The second semester the courses taken by the students are 'Vectors and functions', 'History of mathematics', 'Geometry and combinatorics' and 'Calculus'. Data were collected through observations of lectures and tutorials in the two courses in which functions are studied. To get more specific data on the individual students' use of words, visual mediators, narratives and routines individual questionnaires and interviews were carried out. On three occasions questionnaires were distributed to the students in the tutorials. The two first questionnaires asked the students to explain what a function is and the third asked if given graphs (e.g. $y=1/x$, the line $x=2$, a discrete function) and algebraic expressions (e.g. $f(x)= x^2$ for $x\geq 0$; x for $x<0$) represent functions. Individual interviews were conducted on four occasions with students who volunteered for interviews. The first three interviews took the student's answers in the questionnaire as a starting point. The fourth interview was conducted at the end of the next semester, five months after the third interview. During that semester the students have taken courses in mathematics education, in probability and statistics, and general teacher education courses. The interviews were semi-structured and the duration was about 30 minutes. They were held in a separate room in the math-library on a time chosen by the student and were audio recorded and later transcribed in full. All students were informed about the research and took part voluntarily. None of the authors were involved in the teaching or examinations. The students were informed that the data would not be presented to course leaders and examiners in a manner that would reveal individual identities.

In the pre-calculus course 'Vectors and functions' polynomial and exponential functions are presented. At the end of the course there was a discussion about the definition of function. The definition given in the compendium used is as follow: "A function is a mapping that for all numbers x in a specified set maps the number to another number which is called the value of x for the function and is noted $f(x)$" (personal translation). In a lecture the students worked with an exercise meant to practice reading of mathematical texts. Here a more abstract definition of a function was given: "A function is a subset of the Cartesian product in which all elements x in the domain occur in exactly one pair (x, y)" (personal translation). A discussion about this definition took part in the lecture. In that discussion the lecturer gave an example of a function with a domain consisting of five students and a codomain consisting of marks (A-F). The function was represented by the graph where marks are ordered to each of the students. The lecturer also pointed out that no rule or arithmetic formula was included in that example. In the course 'Calculus' a definition was given in the beginning of the book used: "A function is today understood as a rule or a process that in a well-defined and unique way remake (transform) some specified objects to new objects" (personal translation). However, there was no discussion about the definition of the function concept in the lectures or tutorials. An interpretation of the absence of a formal definition and a discussion can be that this was expected already known by the students. During the course, the derivative was used to analyse functions with respect to extreme values. These analyses together with studies of asymptotes were used to draw graphs.

The qualitative analysis of the questionnaires and interview transcripts was focused on the students' ways of talking about their understandings of function. As mentioned above, we used Sfard's categories to identify changes of the interlocutors' mathematical discourse. Through repeated readings we sorted the utterances in matrixes and produced narratives about the students' development of understanding.

RESULTS

The presentation of the results gives a review of how each of the four students wrote and talked about their understanding of the function concept in the questionnaires and interviews during the year.

Student A

Student A is a woman, about 35 years old. She is an in-service teacher studying math to get certified also as a mathematics teacher in secondary school. In the first questionnaire she wrote that a function is "e.g. a variable a that depends on b" and gives an example: "The prize of a taxi journey depends on how far (s) I go and how long time (t) it takes, $p=as+bt$". The narrative used in the example is an everyday example, but we can also notice that she used the mathematical word variable in her description of a function. She used a visual mediator, the formula for the prize. In the interview she presented narratives including everyday examples of functions and she was using mathematical words in a correct way.

In the middle of the semester student A wrote in the questionnaire that it is a function if "for every x there is a value y". She pointed out that the pairs (x, y) do not need to be connected and she drew a graph with discrete domain. We can notice that the narrative at this time was more formal and included mathematical expressions as 'for every'. She also made a visual mediator by herself, a graph, showing that she was comfortable with using graphs as illustrations. In the interview she said that she did not remember the definition but "it was something with exactly one." She talked about her change of understanding, from functions as rules to accepting collections of points as functions, as an extension of her previous understanding. She used a narrative that is a reflection of the more abstract definition in the exercise mentioned above:

> Then I thought like that you have one rule that somehow determines what is going to happen, that was my relation, and now it feels like, no, there doesn't have to be any relation at all, we just have to say that they come in pairs.

In questionnaire 3 student A answered correctly on which of all the graphs and algebraic formulas that are functions. In the interview she gave a definition of function: "For every value x there is a value y, or in other words, one value x can only give one value y, if you put in one x you will get one y, you can't put in one x and get two y". She said she used this as a routine to decide if the curves are functions or not. She also repeated what she said in the second interview, it does not need to be a rule; if one x gives one y then it is a function. There was no fourth interview since she at that time was back to her work as a teacher.

Student A used *mathematical words* already in the beginning of the semester. Her vocabulary was expanded during the semester and included correctly used formal words. She used formulas and graphs as *visual mediators* but they seemed not to be crucial for her understanding. The *narratives* started in everyday examples and developed to formal expressions, they also changed from function as a rule to function as pairs. She told that her encounter with the abstract definition was important for this change. She got a *routine* for deciding if it is a function or not. The development described resulted in a transformed understanding. The transformation was pretty smooth; she started on a high level of understanding.

Student B

Student B is a man, about 25 years old. In the first questionnaire he wrote that a function is to "from given data describe a course of events depending on different factors". In the interview the narratives were everyday examples as the prize of a taxi journey, but unlike student A he did not use mathematical words in the examples. He talked a lot about visual mediators such as graphs and their interpretations. He could not give any definition of function, nor in the first interview, neither in the second. He wrote in the second questionnaire:

> I regard functions as a relation of dependence where e.g. y depends on x. This can be shown graphically, but not necessarily. A function can express and maybe predict processes from given data.

In the interview he said that he still mentally remains in the upper secondary school where, as he interpreted it, functions are regarded as graphs. However, as also could be seen in the quote from the questionnaire, he now used variables and some mathematical words like 'depends on' in the explanation of what a function is for him. Visual mediators are still important for him, especially graphs but he also mentioned algebraic formulas. He also talked about routines of algebraic manipulations to get the possibility to "see it in the formula". He mentioned the exercise where a more abstract definition was given but the content in that definition was not involved in any narratives. In the third questionnaire he answered wrongly that all the graphs were functions. He said in the interview that he did not have any routine for checking if a graph was a function or not. He had not recognized, or could not make use of, that to each x there needs to be exactly one y. In the narratives we can find small traces of the more abstract definition, but it is not made clear and established. He referred to the lecturer: "He said that even if you just have dots it could be a function". Student B talked about algebraic representations but still he connected functions strongly to graphs: "You feel a bit unsure, but if you can show something graphically then you feel that it is a function". The visual mediators, formulas and especially graphs, were important for his understanding. A semester later, in the fourth interview, student B still could not give any definition. He said that a function is "like a connection, it is something that gives something, and maybe you could draw it graphically."

Looking back on his talking in the first interview it could be said that he still nine months later was talking about functions nearly in the same way. He used just a few *mathematical words*; he mostly used colloquial discourse during the whole study year. He strongly connected functions to *visual mediators*, especially to graphs. The *narratives* did not include any definition of the concept and just small traces from the more abstract definition given in the exercise mentioned above could be recognized. He did not use any *routine* for deciding if a curve is a function. He developed routines for algebraic manipulations but this was not enough to transform the understanding in a way that could be expected during these courses.

Student C

Student C is a woman, about 30 years old. She was taking the courses this semester a second time because she did not pass the first time. In the beginning of the semester she was really happy about following the course with this good lecturer. In the first questionnaire she wrote:

> A function describes a relationship between axes in a coordinate system and a line or a curve in the same coordinate system.

In the interview she talked about a function as a ratio between x and y. The notion $f(x)$ was the first thing she mentioned when asked about functions. She also talked several times about the system of co-ordinates and about curves, but she did not use the word graph. She just used some simple mathematical words and the narratives were strongly connected to visual mediators. In the second questionnaire she wrote: "A function is an expression for conditions between values." In the following interview she could not mention any definition of function but remembered from the lecture: "...well, he said a very good definition a couple of lessons ago but I haven't learnt it by heart so." This is the only instance where traces of the more abstract definition, given in the exercise mentioned above, could be found in her utterances. She did not mention system of co-ordinates and neither did she use the notion $f(x)$. It seemed as she had lost the connection to visual mediators without coming to a good use of more formal narratives and mathematical words. Some narratives involved mathematical words but partly they were used in a wrong way.

From the third questionnaire we can conclude that she had not at this time recognized that to be a function it is necessary that to each x there is exactly one y. She answered wrongly that $x=2$ is a function and did not give any answer on discontinuous graphs. She had no routine to check if a curve is a function or not and the motivations given were in a colloquial discourse and some of them incorrect. Student C did not take part in the third interview. At that time, in the end of the semester, she was really upset and frustrated. She had not passed the examinations and felt that the teachers did not help her. She said that she did not manage to talk about mathematics at all. Five months later, at the end of the next semester, she by herself said that she wanted to take part of the interview. Now she shined of self-confidence and had passed some of the examinations. She said that she now had matured in her understanding of mathematics and no longer was afraid of functions. But still she could not give any definition. She said that she did not grasp the key aspect. In this interview, as in the first, she talked about the notion $f(x)$, the system of co-ordinates and curves, but at this time she also used the word graph.

Summing up her talk during the study year we notice that she started using mostly a colloquial discourse. During the semester she included a few *mathematical words*, sometimes partly used in a wrong way. In the beginning of the semester *visual mediators* were really important for her. After a while it seems like she lost this connection but at the end of the study year she again leaned on visual mediators. The *narratives* were strongly connected to the visual mediators. There was no formal definition and no trace of the more abstract definition. She never got any *routine* for checking if a curve is a function or not. Her development was not enough to transform her understanding in a way that could be expected during these courses.

Student D

Student D is a woman, about 35 years old. Before she started her teacher education she had been working as an economist. Student D is the same student who in an earlier paper was given the fictitious name Kim (Pettersson, 2012). As it was presented in that paper, in the first interview she said that a function is a recipe. However, in her narratives she strongly connected functions to everyday examples and did not give any algebraic examples. In the second interview she said:

> …what's new in my understanding of functions is that there doesn't have to be a rule that defines this relationship; sometimes there's just a relationship between two… chosen things.

She continued with a reference to the lecture where she met the more abstract definition: "... it could be a bit arbitrary like in the example with the marks". She said this example was an eye opener for her, but also that she still could not use it by herself. "I think I'm still like in between two understandings of this and I suppose I'm trying to find a way to put them together." She also pointed out that the important characteristic of a function is that for every x there is one single y. In the third questionnaire she answered correctly on all the graphs. She was using a vertical test checking that it was just one y for each x. In the interview she said that now it is not the functions that is the problem: "...but the functions, I think they've suddenly become, well I don't know, a table, a platform". In interview four she was able to discuss functions in terms of objects and processes and said: "I can sort of move around within this entity".

The journey of understanding for student D started in everyday examples and functions as rules. She was correctly using *mathematical words* from the beginning and expanded her vocabulary during the semester. She did not lean on *visual mediators*. The *narratives* started with function as a rule and ended up with function as pairs without need for a rule. She also became able to discuss functions as objects. She got a *routine* for deciding if a graph is a function or not. She talked about her encounter with the more abstract definition and how it opened her eyes for a new understanding. Her new understanding merged together with the old one and she got a new unified and transformed understanding of the threshold concept of function.

CONCLUSIONS AND DISCUSSION

The two students, A and D, who got a transformed understanding of the threshold concept of function, expanded their use of mathematical words. They also changed their narratives from everyday examples and function as a rule to function as pairs. The students' encounters with the more abstract definition seemed to be important for this transformation of their understanding. These students also got a routine for deciding if a curve represents a function or not. They used graphs and formulas but the visual mediators were not crucial for their understanding. The other two students, B and C, did not develop their understanding enough to get a transformed understanding of the threshold concept of function. Through the whole study year they just used a few mathematical words. They strongly leaned on visual mediators; curves in co-ordinate systems and interpretations of graphs were important for these students. The narratives were strongly connected to visual mediators and were given in colloquial vocabulary. No formal definition was included in the narratives and no, or just a small, trace of the more abstract definition could be found. These students did not have any routine for deciding if a curve represents a function or not.

The results show the complexity involved in developing the understanding of the threshold concept of function. For these four students it was important to expand their use of mathematical words. It seemed like the need of precision requires the use of mathematical words. Visual mediators are surely important for students' understandings but what could be seen from the development of these four students is that it is not enough to stay within an understanding strongly leaning on the visual. For two of the students, A and D, the encounter with the more abstract definition seemed to have had an important influence in the development of their understanding towards a more formal level. When the students' view of a function as a rule developed to a function as a set of pairs, they understood also discontinuous and discrete functions. It also helped these students to understand the key idea that for each x there should be exactly one y.

The results of the study substantiate the importance of expanding the mathematical vocabulary and developing a balanced use of visual mediators and routines. Changing the discourse to include more formal narratives also supports the transforming of the understanding. Through this longitudinal study the complexity in coming to understand a threshold concept has been made visible. Since threshold concepts are crucial in students' development of their understanding of mathematics it is important for university teachers to know how to support this transformation (Meyer & Land, 2005). An early development of a solid understanding of functions may also make the transition from secondary level to university easier for the students (Thomas et al., 2012). The transition of function as a rule to function as pairs is maybe related to the, for students often hard (Viirman et al., 2010), reification of function as a process to function as an object. This hypothesis needs to be tested in subsequently research. To raise the level of awareness among university teachers in mathematics about these things can be regarded as a didactical implication of our study.

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