THE TRANSITION FROM UNIVERSITY TO HIGH SCHOOL AND THE CASE OF EXPONENTIAL FUNCTIONS

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University mathematics programmes cater, in various ways, to future teachers of mathematics. The gap between the university contents and methods on one hand, and on the other hand secondary school mathematics, is investigated in studies of how students experience the beginnings of university mathematics programmes. Here we consider the gap as it appears at the other end of the students' experience at university, as they try to relate their new mathematical knowledge to contents and methods in secondary school teaching. The main points of the paper are to introduce a theoretical model to analyse the changes in relation to mathematical knowledge which may be operated at that point, and to show its use on a significant example (the construction of exponents or, more generally, of the exponential function).

AN DOUBLE TRANSITION PROBLEM

A large part of the existing research on university mathematics education is devoted to the study of the specific challenges students face at the beginning of a study programme with a strong or exclusive mathematical component (see Gueudet, 2008 for a review of such research, as well as several papers in the CERME7 version of this working group). These challenges are *institutional transition problems* since they arise as students change institution, and several challenges appear to be related to major differences between emphases and methods of teaching in high school and university. At the opposite end of the university programme, the students face another type of institutional transition, namely that from academic mathematics to various professions. In this paper we are interested in the nature of this second kind of institutional transition, and in particular how a mathematics programme may seek to "smoothen" it in view of the profession of secondary mathematics teacher.

Already Klein (1908, p. 1) observed that students taking this direction may face a "double discontinuity" as they move from high school to university, then back again:

> At the beginning of his studies, the young student is faced with problems that in no way remind him of the [mathematical] things he worked with in school; naturally he then forgets these matters quickly and thoroughly. If he becomes a teacher after having finished his studies, he must suddenly teach this time honoured elementary mathematics in a school like fashion; and as he cannot by himself see the connection between this task and university mathematics (...) his university studies become just a more or less pleasant memory which has no influence on his teaching. (Translated by the author)

Indeed in many countries, secondary school teachers receive their mathematical education in study programmes geared towards academic "pure" mathematics. The second discontinuity or gap, from university mathematics to secondary school mathematics, appears as a problem to the extent the students' outcome from the programme does not function as a resource for teaching mathematics at the secondary level. While there is extensive research on how teachers experience the transition from teacher education into teaching practice (see e.g. Winsløw, 2009, for an overview), there remains a practical need for systematic didactical research on how standard undergraduate mathematics is, or could be, developed in view of facilitating its use by students in inquiries related directly to high school mathematics.

This problem is clearly at the borderline between research on mathematics teacher education and research on university mathematics education. But it is of relevance to the latter for at least two reasons: university mathematics programmes often include courses specifically designed to treat elementary mathematics "from a higher viewpoint", as Klein (1908) put it in his book title; and also one may speculate that the two "smoothing problems" might usefully be approached together.

In this paper, we present a new theoretical model of the second transition, with a concrete case to illustrate it. We first introduce the institutional context in which we have considered the problem (a capstone course). Then we explain the theoretical model in order to situate this somewhat arbitrary context within the general problem of institutional transition and the corresponding changes in relation to mathematical knowledge. Finally, we explore a case which is both illustrative and important in its own right, namely the work with exponential functions in a capstone course.

We emphasize that the primary aim of this paper is to present the problem (of transition from *U* to *HS*) and our theoretical model of it, while the case serves to illustrate both through a concrete didactic design, its premises, and some observations of its effects. The case is not an empirical study; it helps to substantiate the model.

CAPSTONE COURSES AND THEIR COUSINS

Universities operate with different traditions and methods across the world. In American colleges and universities, there is a long standing tradition of "senior year programs", which include the so-called "capstone courses":

> The capstone course typically is defined as a crowning course or experience coming at the end of a sequence of courses with the specific objective of integrating a body of relatively fragmented knowledge into a unified whole. As a rite of passage, this course provides an experience through which undergraduate students both look back over their undergraduate curriculum in an effort to make sense of that experience and look forward to a life by building on that experience (Durel, 1993, 223).

In mathematics programmes, capstone courses are not always aimed at preparing students to a life as mathematics teacher, but many are, as evidenced by the offer of text books such as Usiskin, Peressini, Machisotto and Stanley (2003) or Sultan and Artzt (2011). These texts contain exposition and exercises related to critical areas of secondary mathematics such as trigonometry, number systems or plane geometry, but from the "higher viewpoint" of academic mathematics, just as in Klein's (1908) treatise which had similar goals (for its time). In fact, a related feature of the German

Stoffdidaktik (content didactics) tradition is to make use of academic mathematics in the development and study of secondary school mathematics. One could mention also the *concours* system, a kind of preservice teacher examination in France and other Romance countries, as an example of transition measures of this kind which are more or less integral parts of the university curriculum within a discipline.

Our context

At the University of Copenhagen, almost all mathematics courses focus on inducing students into the methods and theory of the academic disciplines in question. Only a minority (~40%) of the students become teachers, and many students are undecided about career plans during most of their study. Even the options to specialize in the history or didactics of mathematics (at the master level) are aiming to develop familiarity with research and its methods, rather than professional skills as a teacher.

It should be noted here that to become a high school teacher in Denmark, one needs to study two disciplines, normally a major (for 3-4 years) and a minor (1½-2 years). Students who do a minor in mathematics get to study the first parts of the bachelor programme in mathematics, including abstract algebra, analysis and differential geometry. For some of these students, that material appears both relatively advanced and rather disconnected from what they perceive as relevant to teach mathematics in high school. Then, a few years ago, a course on "Mathematics in a teaching context" (UVmat) was introduced as an option at the bachelor level, primarily in view of students who take mathematics as a minor (but open to others interested in teaching). In recent years it was co-taught by the author and a colleague; in the version of 2011-2012 it was based on Sultan & Artzt (2011). The course has gradually adopted the idea of serving as a capstone course with the following aims:

- Identifying and filling serious gaps in the students' knowledge of select areas high school mathematics which remain after taking more advanced courses;
- Treating selected topics – mainly from high school analysis and algebra – but from "from a higher viewpoint" (cf. Klein, 1908), in an attempt to illustrate how the material learned in university can serve also when working with more elementary topics in view of teaching.

To sharpen the meaning of these points further, we introduce some elements of the anthropological theory of didactics as well as a more general model of the double transition mentioned in the introduction.

THEORETICAL MODEL

It is a basic contention in the anthropological theory of didactics (Chevallard, 1991, 206-207) that the relation of an individual to an object o of knowledge is strongly conditioned by the institution I in which this object of knowledge lives. For a didactic analysis, we usually consider the individual as occupying a position p within the institution, for instance as a first year student, and study the relation $R_I(p, o)$ of this position in the institution to o, where the subscript stresses that p and o depend on I.

This abstraction is by no means artificial. In fact, data may inform us, with more or less precision, about the relation of smaller number of student to the practice and theory related to Taylor series, or a similar object of knowledge o that of course must be more precisely delimited. But our real aim is usually more general, such as investigating the relation to o of first year students in the University of Copenhagen – or perhaps a generic university of a similar kind.

Such a relation can, indeed, exhibit many variations, which are not exhausted by simple measures of "mastery" of target knowledge. The relation involves also, for example, the *status* assigned to o with respect to other knowledge objects, which may depend highly on p and therefore also on the institution I. For instance, to Spanish high school students, *limits* of functions may end up with an exclusive and marginal status as a formal preliminary to defining *derivatives* of a function (Barbé et al., 2005). Another important variable in the relation is the *modality of access* which I enables p to have to o, for instance, solving exercises, study of a textbook, autonomous inquiry etc.

To model the "objects of knowledge", Chevallard (1999) introduced the notion of *praxeology* (do read Barbé et al., 2005 if you are unfamiliar with this). In this model, an object of knowledge o is modeled as praxeologies which consist of two related parts: *practical knowledge* (abbreviated P) such as a method to estimate the error of a given Taylor approximation of a given function, at a given point; and *theoretical knowledge* T, such as a way to explain and justify P. In fact, the knowledge object o which we consider may be formed by a whole collection of praxeologies (P, T).

When considering the double transition outlined in the previous section we are actually investigating transitions of the type $R_{HS}(s,o) \rightarrow R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$ where the institutions are that of a more or less well defined high school (HS) and university U; also, s and σ refer to more or less delimited positions as student in these institutions, and t to the position as teacher in HS. Finally o and ω are knowledge objects of a more or less comparable nature. Chevallard (1991) developed the notion of didactic transposition to explain how a knowledge object is transformed in view of enabling students in an institution such as U or HS to establish relations with it; an important case is the transposition of praxeologies ω developed and taught at universities into praxeologies o taught in schools. The second transition above is, clearly, of another nature, and takes place at another pace; still, the success of the didactic transposition certainly depends on the relation $R_{HS}(t,o)$ and this, in turn, rely in part on a past relation $R_U(\sigma,\omega)$ with the transposed object ω, and with teachers now in position t after having been in the position σ.

In capstone courses, we focus on students in position σ and consider how $R_U(\sigma,\omega)$ could be developed in view of a future situation of the students in a position t, and we try to prepare bridges between relations of type $R_U(\sigma,\omega)$ and a more or less hypothetical relation of type $R_{HS}(t,o)$. As a matter of fact, the relation may not be hypothetical at all. Due to an increasing shortage of graduates to fill positions in

Danish HS, some of the students in UVmat are already part time high school teachers. This clearly is a special situation worth particular attention as these students are already experiential in some of the challenges transitions of type $R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$ which are our topic here. However, we do not focus on this situation here.

CASE STUDY: EXPONENTS AND EXPONENTIAL FUNCTIONS

We now consider, as knowledge object o, the approaches to exponents a^b currently or potentially found in Danish high school (HS), and we consider the set of knowledge objects ω more or less close to o, which are taught in the bachelor programme in mathematics at the University of Copenhagen (U). Our focus is on how a capstone course such as UVmat may contribute to (and draw on) $R_U(\sigma,\omega)$ in order to prepare the transition to the relation $R_{HS}(t,o)$.

High school approaches to exponentiation

There is a certain variation in the approaches to exponentiation found in text books and, conceivably, therefore in the relation students and teachers in *HS* will develop to o. Most textbooks define a^x for rational x, using more or less elaborate justifications of the formula $a^{m/n} = \sqrt[n]{a^m}$ based on the definitions of $a^{1/n}$ and a^n. Given that real numbers and limits are not rigorously treated in HS, the variation lies in how the books explain the passage to real exponents. Here are some typical examples:

> The power is calculated by approximating the exponent by a finite decimal number. How many decimals you include depend on the required accuracy (Timm & Svendsen, 2005, 26; translated from Danish by the author)

> In Chapter 3 we saw how to calculate powers where the exponent is integer and positive, 0, integer and negative, and rational (fraction). Strictly speaking we have not explained the meaning of a symbol like $7^{\sqrt{3}}$ but we assume CAS will take care of this. (Carstensen, Frandsen & Studsgaard, 2006, 82; translated from Danish by the author)

> It is also possible to extend the notion of power to the case of arbitrary exponents like for example π and $\sqrt{11}$ but it will take us too far to do that here (Brydensholt & Ebbesen, 28; translated from Danish by the author)

After this, all books operate with exponential functions as functions defined on ℝ. There is nothing surprising about this and one should not overestimate the impact on $R_{HS}(s,o)$ of the differences like the above. Judging from informal questioning at lectures over the past few years, few students recall the definition of powers with rational exponents, and even less wondered how to define a^x properly and in general.

Exponentiation in the mandatory bachelor curriculum

All texts used in U simply *assume* the existence of exponential functions as part of the prerequisites from HS. Students are also assumed to know certain basic properties of exponential functions, such as continuity, derivatives and other specific rules, like $a^{x+y}=a^x a^y$ for $a \geq 0$ and x, y real. Exponential functions appear frequently in the first courses on calculus and analysis – for instance, in examples of calculating Taylor series, as solutions to differential equations, as building blocks in functions of several variables and complex functions, as a tool to study complex numbers (polar form) or functions, etc. Briefly speaking, in $R_U(\sigma,\omega)$ the meaning and basic properties of a^x (for real *a, x*) are certainly available, but they are neither questioned nor explained.

Exponentiation in UVmat

There are several classical ways to approach the construction of a^x for $a>0$ and x real, some of which are elegant but out of reach in the first year of HS when this object is first encountered. In UVMat, it is a specific point to develop $R_U(\sigma,\omega)$ so as to know and relate five of these, based on

(1) "Direct" construction, starting with the case $x\in\mathbb{N}$ and extending first to $\mathbb{Q}$, then to $\mathbb{R}$;

(2) The inverse function to $\log_e$, itself constructed via $\log_e(x) = \int_1^x \frac{dt}{t}$;

(3) The initial value problem $dy/dx = y,\ y(0) = 1$;

(4) The functional equation or "property" $f(x+y) = f(x)f(y)$;

(5) The power series $\sum_{k=0}^{\infty}\frac{x^n}{n!}$.

Here, (4) can be used to show uniqueness of a certain form, detailed in the next subsection, while existence must be based on other methods. We notice that (1) is similar to the brief explanations found in *o* (first year) as explained above, but it still relies on deeper properties of $\mathbb{R}$ that are may be identified in ω but not in *o*, while (2)-(5) are all entirely beyond the scope of *o*.

Indeed, giving a complete account of (1) is somewhat challenging even based on $R_U(\sigma,\omega)$. The textbook by Sultan and Artzt (2011, pp. 242-250) provides such an explanation up to rational exponents. For the last step, it provides an insufficient "proof" while acknowledging that "there are some serious issues with this, and to get into all of them would be beyond the scope of this book" (p. 250). The serious issues concern the equation $\lim a^{q_n} = a^{\lim q_n}$ for a sequence (q_n) of rational numbers, assumed in the proof with no warranty even of the existence of the first limit and its independence of a choice of sequence (q_n) tending to a given real number. While one may argue that this could be sufficient for a high school teacher, $R_U(\sigma,\omega)$ can be developed to give a complete explanation by making use of the following property of $\mathbb{R}$, taught in first year and equivalent to the completeness property : every ascending sequence of real numbers with an upper bound, has a limit.

From a didactic viewpoint, the real challenge lies of course in the effective extension of $R_U(\sigma,\omega)$ and its possible consequences for $R_{HS}(t,o)$. In a recent version of the course (2011), we chose to present (1), (2) and (5) in lectures, while leaving the approaches (3) and (4), and the links between them, to an exercise. We now take a closer look at the tasks left to students and the outcomes.

Students' turn

The 25 students worked on the following exercise (part of a weekly assignment for group work; translated from Danish and slightly rephrased to be self-contained):

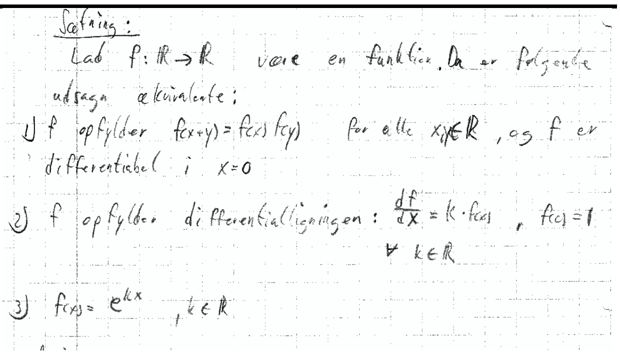
a) Show that if a function $f: \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ then $f(0) = 1$ or f is the zero function. Give a (non-zero) example of such a function.
b) Show that if a function as in a) is differentiable in 0, then it is differentiable on all of $\mathbb{R}$. [Hint: look at $(f(x+h)-f(x))/h$ for $h \neq 0$.] What is the derivative of f?
c) You know from calculus that the initial value problem $dy/dx = ky$, $y(0) = 1$ has the unique solution $y = e^{kx}$. Use this together with the results obtained in a) and b) to provide a characterization of exponential functions.

The point in b) is that the condition implies that f' exists and equals $f'(0) \cdot f$ in all cases. It is the *existence* of a solution, mentioned in c), which the "real meat" of the result assumed, as a special case of Picard's theorem which is treated in first year calculus. The uniqueness, which is the main point here, is quite easy to prove.

The first questions are technical questions and most groups were able to solve them on their own, with a few minor lapses like forgetting the case $f = 0$ in b). However, the third point proved to be challenging for almost all groups. The challenge seems to be the word "characterization" and also to link the form e^{kx} to the notation a^x privileged by the text book. In fact, the theoretical point of view involved with recognizing and formulating a "theorem" is not frequently required from students in $R_U(\sigma,\omega)$ at least in the work prior to UVmat. On the other hand, with the result recalled from calculus and the results proved before, there is – from a technical point of view – a small step to realize and prove that a function f is an exponential function (i.e. $f(x) = a^x$ for some $a \geq 0$) if and only if f is differentiable at 0 and satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Of course there are other possible formulations, and in this case we made a point of leaving the formulation open so that alternatives could appear (in fact they did). For instance, the term "exponential functions" could be defined as including the zero function, or not.

The students have access to supervision during their work, and many needed help to even get started on part c) of the task. Looking closer at students' final formulations of theorems, we find a number of shortcomings that are of a more logical nature. Here is one example (translation to the left, the original in Danish to the right):

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ be a function. Then the following propositions are equivalent:
1) f satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and f is differentiable at $x = 0$
2) f satisfies the differential equation
$df/dx = k \cdot f(x)$, $f(0) = 1$
$\forall k \in \mathbb{R}$
3) $f(x) = e^{kx}$, $k \in \mathbb{R}$


Sætning:
Lad $f: \mathbb{R} \to \mathbb{R}$ være en funktion. Da er følgende udsagn ækvivalente:
1) f opfylder $f(x+y) = f(x) f(y)$ for alle $x, y \in \mathbb{R}$, og f er differentiabel i $x = 0$
2) f opfylder differentialligningen: $\frac{df}{dx} = k \cdot f(x)$, $f(0) = 1$
$\forall k \in \mathbb{R}$
3) $f(x) = e^{kx}$, $k \in \mathbb{R}$

For instance, we note the strange appearance of the letter k in part 2) and 3). In 2) an existential quantifier (∃) would be more relevant than the universal (∀), while in 3), the unspecified status of the letter k is just as problematic. On the other hand, it appears from the students' proof that these points are really mainly of a formal nature. Indeed, many students produced essentially sound proofs despite occasional lacks of clarity in their statements, as above. For these students, we can focus on the formal features of mathematical expression which are, of course, of special importance to $R_{HS}(t,o)$ regardless of the piece of knowledge o involved, as the teacher should not only be able to express formal relations with a variation of formality while retaining basic correctness, but should also be capable of assessing less correct ones.

For other groups of students – representing roughly a quarter of the 25 students – there appear to be real challenges even in the parts of the reasoning that are completely within the kind of arguments that appear in HS text books. As an example, one group did not only formulate a "Theorem" which is formally incorrect, but also provided a proof that is flawed from the first lines, as the following excerpt (translated into English) shows:

> **Theorem:** An arbitrary function $f : \mathbb{R}\to\mathbb{R}$ which is differentiable in $f(0)$ and satisfies $f(x+y) = f(x)f(y), f(0) \neq 0$ for all $x, y\in\mathbb{R}$, is of the form e^{kx}
>
> **Proof:** A function f satisfying $f(x+y) = f(x)f(y)$ could be the exponential function. We therefore let $f(x) = e^{kx}$, $f(y) = e^{ky}$ in the rest of the proof. (…) f also satisfies $f(0) = 1$ since this is well known for the exponential function and can be easily seen by:
>
> If $y = -x \Rightarrow f(x-x) = e^{k(x-x)} = e^{k\cdot 0} = 1 \neq 0$
>
> (…) [*The group goes on to show, by a lengthy calculation repeating parts of the argument from b) in the special case f(x) =* e^{kx}*, that f satisfies the differential equation df/dx = kf. They conclude as follows*]
>
> From Picard's theorem we know that for all real numbers k there is one and only one solution to $dy/dx = ky$ satisfying $y(0) = 1$. As we have shown the exponential function satisfies this we know that the exponential function is the only one satisfying this, and the theorem is proved.

For students who produce "proofs" like this one, even formulating autonomously a proof within the core contents of o is likely to be a serious challenge, and one could say that the task c) is perhaps missing the level at which they need to improve their relation to basic HS knowledge. Fortunately, UVmat also offers many much simpler tasks and discussions, focusing more directly on o, such as a) and b), which in fact were solved almost perfectly by all groups, including those who failed on c).

PERSPECTIVES AND CONCLUSION

All of the five approaches (1)-(5) involve technical work that is to some extent beyond $R_{HS}(s,o)$; and as power series are not treated in HS, (5) is perhaps of little significant importance to $R_{HS}(t,o)$. The other four can strengthen $R_{HS}(t,o)$ in the sense of providing extensions or alternatives to standard presentations. In fact, the above tasks do not go outside the formal boundaries of the HS curriculum which does include a formal definition of derivative as well as (at the most advanced level) simple differential equations. The main point of including in $R_{HS}(t,o)$ alertness to the non-triviality of o from a mathematical point of view – which is, more or less, absent in $R_U(\sigma,\omega)$ – could of course be achieved just by working on (1). However, to include alternatives such as (2), (3) and (4), which are strongly related to more advanced parts of the HS curriculum, is clearly of merit to $R_{HS}(t,o)$ as well. For instance, a rigorous approach to integrals has been part of the HS curriculum and might become part of it again, with the increasing use of computer algebra systems to take care of calculations; and then (2) would be a reasonable and elegant way to make up for the shortcomings of o based on (1), including the ease with which fundamental properties of exponential functions may be derived using this construction.

It is interesting to reflect on the difficulty which students had with c) even after solving a) and b) correctly. This exemplifies a phenomenon which we have noticed also in regular, semi-advanced analysis courses, namely a transition which occurs *within* U rather than as a result of changing institution. These transitions have to do with the kind of relation $R_U(\sigma,\omega)$ students need to develop to a university situated mathematical praxeology ω. In the terminology of Winsløw (2008) the transition is said to be of type I if theory is to be used and operated with (by students); it is said to be of type II when tasks require autonomous development of theory.

In a) and b) students are faced with relative precise tasks, which – although they do involve the theoretical level of ω – can be carried using familiar techniques (operations with the definition of derivative etc.). However, the task c) is less "precise" and students have no familiar technique to solve it, namely to formulate and prove a "characterization of exponential functions". This kind of task is indeed relevant to the position $R_{HS}(t,o)$ of a teacher relative to a piece of knowledge o, such as explaining the sense in which a property (to be precisely defined, as part of the task!) characterizes a class of mathematical objects (equally to be defined). Thus, while the relation $R_U(\sigma,\omega)$ necessary to solve a) and b) is already different from $R_{HS}(s,o)$, this is not so much due to a difference between o and ω, but because of the fact that theory (definitions, proof based on limit operations) has to be drawn upon by σ but usually not by s. On the other hand, going from b) to c) represents an instance of the transition $R_U(\sigma,\omega) \rightarrow R_U^*(\sigma,\omega)$ where the asterisk indicates autonomy with respect to the theoretical level of ω; notice that this is a transition of type II in the sense of Winsløw (2008). We argue that $R_U^*(\sigma,\omega)$ is more relevant for a "mature" teacher relation $R_{HS}(t,o)$ than $R_U(\sigma,\omega)$. While a capstone course may have other agendas as well, this is an important point in many of the tasks proposed to students in UVmat, the main difference from other mathematics courses in the programme being that ω remains elementary and close to a corresponding o in HS.

By way of conclusion, we hypothesize that in general, the second institutional transition $R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$ referred to in the introduction is not only very different from the first institutional transition $R_{HS}(s,o) \rightarrow R_U(\sigma,\omega)$, it is also closely linked to institutionally internal transitions of type II within the university programme (in the above notation, $R_U(\sigma,\omega) \rightarrow R_U^*(\sigma,\omega)$). These are in general very hard to achieve (cf. Winsløw, 2008). The aim of a capstone course could be – among other things – to achieve it, not in general, but for as selection of mathematical praxeologies ω close (if not identical) to praxeologies *o* which are, or could be, developed in high school. In short, a capstone course could strive deliberately to get students started in autonomous work with theoretical parts of the academic mathematical praxeologies ω that are most directly related to school mathematics praxeologies *o.*

REFERENCES

Barbé, J., Bosch, M., Espinoza, L. and Gascón, J. (2005). Didactic restrictions on teachers' practice – the case of limits of functions in Spanish high schools. *Educational Studies in Mathematics* 59, 235-268.

Brydensholt, M. & Ebbesen, G. (2010): *Lærebog i matematik, 1*. Aarhus : Systime.

Carstensen, J., Frandsen, J. & Studsgaard, J. (2006). *MatC til B stx.* Aarhus : Systime.

Chevallard, Y. (1991). *La transposition didactique. Du savoir savant au savoir enseigné*, 2nd edition. Grenoble: La Pensée Sauvage Éd.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique des mathématiques* 19 (2), 221-265.

Durel, R. J. (1993). The capstone course: A rite of passage. *Teaching Sociology* 21, 223-225.

Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67 (3), 237-254.

Klein, F. (1908). *Elementarmathematik vom höheren Standpunkte aus, I.* Leipzig: B.G. Teubner. Quote here refer to the fourth edition (1933), Berlin: Springer.

Sultan, A. and Artzt, A. (2011). *The mathematics that every secondary math teacher needs to know.* New York and London: Routledge.

Timm, U. & Svendsen, T. (2005). *Matematik C bogen.* Aarhus : Systime.

Usiskin, Z., Peressini, A., Marchisotto, E., and Stanley, D. (2003). *Mathematics for high school teachers: an advanced perspective.* New Jersey: Pearson Ed.

Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle. In A. Rouchier & I. Bloch (Eds) *Actes de la XIIIème école d'été en didactique des mathématiques* (cd). Grenoble : La Pensée Sauvage.

Winsløw, C. (2009). First years of teaching. In R. Evens and D. Ball (eds), *The Professional Education and Development of Teachers of Mathematics*, pp. 93-101. New York: Springer US.