A FRAMEWORK FOR EXAMINING STUDENT LEARNING OF MATHEMATICS: TASKS USING TECHNOLOGY

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This paper, taken directly from the author's doctoral thesis, (Joubert, 2007) develops a theoretical and methodological framing for examining student learning in the context of mathematics classrooms where computers are used. The framing, drawing particularly on the theories of Brousseau (1997), takes into account not only the student interactions with the environment but also the crucial role played by the feedback from the computer. This approach focuses on the processes in which the students are engaged and suggests the sorts of interactions that might provide evidence of student mathematical learning. The paper concludes with a section which analyses an episode of student mathematical activity using this framing.

Keywords: modes of production, computer, feedback, task, graphing

BACKGROUND AND CONTEXT

Mathematical activity, like all human activity, is mediated by tools, which include not only symbol systems, mental representations, algorithms and representational systems (such as functions and co-ordinate graphs) but also physical tools such as pen and paper, calculators, measuring devices and so on. Computers can be seen as a particularly interesting tool because of their 'intrinsically cognitive character' (Balacheff & Kaput, 1996, p 469). For some software used in the teaching and learning of mathematics, the implication is that, to a greater or lesser extent, the software can perform mathematical processes (or 'do the mathematics') (Hoyles & Noss, 2003; Sutherland, 2007) for the user. For example, in traditional mathematics classrooms producing a bar chart from a set of data is seen as a valid mathematical activity, but a data handling package is able to 'do' this mathematics (as described by Ruthven and Hennessey (2002)). This has important implications for the design of classroom mathematical tasks.

A second important characteristic of software, related to its cognitive character, is the fact that software provides feedback for the user;

> 'The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner input, and the feedback of the environment is provided in the proper register…' (Balacheff & Kaput, 1996, p 470).

This feedback, together with the ability of the software to 'do the mathematics' as described above, are perhaps the most compelling reasons for seeking to understand the way these tools are used in ordinary classrooms.

The current literature focuses more on teachers' perceptions of the actual use of computers in mathematics classrooms (Monaghan, 2004, Ruthven & Hennessy, 2002) than on the student processes (Lagrange et al., 2003). However, as Tall (1995) argued over fifteen years ago, there is a need to focus our attention on the students' thinking processes in situations where computers have become established in classrooms, to find out 'what is *really* happening under the surface'(p 11, italics in original). It seems that this research agenda has still not been fully met (Lagrange et al., 2003) and that there is still a need to develop a detailed understanding of the students' mathematical activity and learning in these authentic situations.

The data analysed in this paper is taken from a study which aimed to address this research agenda. The study aimed to contribute to the body of research conducted in a naturalistic research paradigm with a particular interest in the use of computer software in mathematics teaching and learning. The paper provides a theoretical and methodological framing for the observation of student interactions in these contexts with the overall aims of a) establishing an understanding of the relationship between observable student mathematical activity and their mathematical learning and b) understanding the role of the computer. It lays out the framing in detail and then provides an example to demonstrate how it has been used.

THEORETICAL FRAMING

To theorise the classroom situation, the paper draws on the theory of *Didactique* (Brousseau, 1997) which identifies the 'didactical contract' between teachers and students and is based on detailed observations of authentic mathematics classroom settings. (This is explained in detail in my doctoral thesis (Joubert, 2007)). Brousseau uses the notion of the *milieu*; 'everything that acts on the student or that she acts on' (p 9) to describe these settings. It is through interacting with the *milieu* and with the tools of the *milieu* (including the task set for them by the teacher) that the students engage with classroom mathematics. The student interactions can be conceptualised as a 'dialogue' between the student (or group of students) and the feedback from the *milieu* (Brousseau, 1997). The feedback from the *milieu* can take many forms; for example verbal feedback from other students and the teacher, an outcome of a game or a graph produced by computer software. The importance of feedback should not be underestimated; as pointed out by Balacheff (1990) :

> 'The pupils' behaviour and the type of control the pupils exert on the solution they produce strongly depends on the feedback given during the situation. If there is no feedback, then the pupils' cognitive activity is different from what it could be in a situation in which the falsity of the solution could have serious consequences' (p 260).

Student activity and mathematical learning

Brousseau (1997) uses the notion of 'modes of production' to describe the different types of dialectic interactions between students and the *milieu*; he suggests that as they work through a mathematical task, they will engage in all or some of the dialectics of action, formulation and validation.

Brousseau (1997) describes a dialectic of **action** as the student constructing an initial solution to the problem straight away, informed by her current knowledge. He explains that, in dialectics of action, students use ‘implicit models’, making decisions based on rules and relationships of which they may not yet be conscious; he suggests that the strategies the student uses ‘are, in a way, propositions confirmed or invalidated by experimentation in a sort of dialogue with the situation’ (p 9).

It is possible that *all* dialectics between the student and the *milieu* in a given didactical situation are dialectics of action; if, for example, the student knows what to do and how to do it in order to complete the task. This would mean that, although she completes the task, she does not need to extend her mathematical knowledge or understanding to do so; ‘simple familiarity, even active familiarity … never suffices to provoke a mathematization’ (Brousseau, 1997 p 211).

There may be an argument that, in some cases of situations of action, the feedback from the *milieu* seems to have little or no role. For example, in a lesson where the student works through a set of examples, it could be that the only feedback they receive is when the teacher reads out the answers. However, as Brousseau argues, the student can be seen to be anticipating the results of her strategies, and in this sense the *milieu* provides feedback, which can perhaps be seen as unrequested and as expected; it does not require the student to adapt her strategies. On the other hand, feedback may occur from time to time as the student works. For example, students may use self-checking methods such as multiplying out factorised functions. In these cases, the dialectical nature of the student interactions is clearer, and the feedback from the *milieu* can be seen as requested and as ‘a positive or negative sanction relative to her action’ (Brousseau, 1997). A negative sanction might prompt the students to formulate new strategies, but it may also result in a sort of ‘guessing’ behaviour, where they simply try something different but do not use the feedback to inform the guess.

Dialectics of **formulation** occur when students meet a difficulty or problem as they engage in mathematical activity; Brousseau explains that when a solution to a problem is inappropriate, the situation should feed back to the students in some way, perhaps by providing a new situation. That means that the student may become conscious of her strategies and begin to make suggestions. Brousseau includes in this category ‘classifying orders, questions etc….’ (p 61). He goes on to say that in these communications students do not ‘expect to be contradicted or called upon to verify … information’ (p 61). In making these formulations the students construct and acquire explicit models and language, which, as Christiansen and Walther (1986) argue, serves to make the learner conscious of strategies: ‘actions become conscious for the learner’ (p 268).

In the discussion above (‘Action’), the possibility of trial and error cycles of student behaviour was proposed. In these trial and error dialectics, the role of the feedback was seen to be only to inform the student that the strategy she had tried was incorrect. However, depending on the nature of the problem, it is possible that the feedback may also provide some clue for the student about how to improve her strategy and she may formulate a new strategy; this approach can be seen as ‘trial and improvement’ or ‘trial and refinement’ approach (Sutherland, 2007).

Both action and formulation involve manipulating ‘moves in the game’ or mathematical objects; **validation** however involves manipulating ‘statements about the moves’ (Sierpinska, 2000, p 6). Validation therefore takes place when an interaction intentionally includes an element of proof, theorem or explanation and is treated thus by the interaction partner (or interlocutor) ‘this means that the interlocutor must be able to provide feedback…’ (Brousseau, 1997 p 16). Brousseau argues that this interaction should be seen as a dialectic because of the presence of the interlocutor. Examples of dialectics of validation include justification (perhaps of a procedure, a word, a language or a model), organising theoretical notions, ‘axiomization’ (ibid p 216), and developing proofs.

Brousseau, while suggesting that all three modes of production are ‘expected from students’, (ibid p 62) argues that it is through situations of validation that genuine mathematical activities take place in the classroom. There seems to be general agreement with this within mathematics education (for example, see Lakatos, Worrall, & Zahar, 1976; Romberg & Kaput, 1999).

Brousseau suggests that situations of validation do not occur very often and are unlikely to occur spontaneously and it is probable that validation will not take place unless it is explicitly called for.

A final comment in this section about validation concerns the role of feedback. The implication from Brousseau’s ‘interlocutor’ (above) is that this interlocutor provides feedback. It is in discussion with this interlocutor that the individual develops his or her arguments; it is unlikely that feedback will come from any source in the *milieu* other than classmates or the teacher because of the need to convince someone else.

METHODOLOGICAL FRAMING

The focus of the investigation was on what takes place in authentic classrooms. The implication is that the classroom situations should reflect, as far as possible, the everyday practice of teachers and students; teachers should teach and students should react as they normally do, all the teachers’ teaching decisions are their own; the choice of topic, software, approach and task. One possible constraint required by the research was that the students should work in small groups or pairs so that interactions between them (what they said and did) could be observed. This constraint is not a major concern; it is common for students to work in pairs in computer rooms.

The teachers taking part in the study were asked to choose one small group of students as a ‘focus group’; the request was that they chose a group whom they perceived to be of average attainment and who might be expected to talk as they worked (to provide verbal data). When the students worked on the task the teacher set, this focus group was observed.

Collecting and analysing data

The focus on processes required the researcher to be able to see and hear what the students did and said as they engaged with the activity, and these were captured on video. Screen activity was recorded using software which 'grabbed' the screen automatically every 30 seconds.

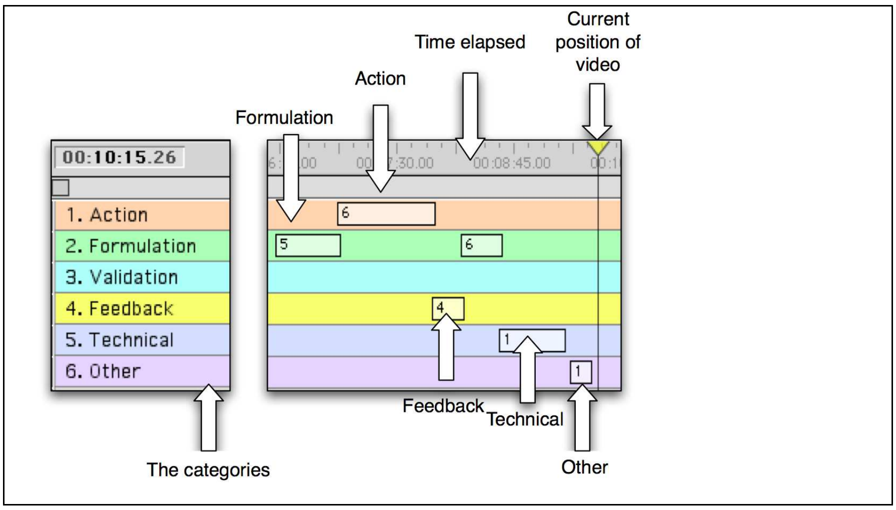
The perspective adopted uses the interactions between students and the setting or *milieu* (in other words, the dialectics) as the unit of analysis, and pays attention to the 'flux of ongoing activity' (Nardi, 1996). However, also as argued above, each interaction is situated in time, and related to all the other interactions taking place as the students work. The implication of this point of view is that, to make sense of interactions, it is necessary to understand the unfolding, or narrative, of all these interactions.

Creating the narrative

The interactions were coded using a scheme derived from the theoretical arguments developed above; to represent the type of student mathematical interactions (action, formulation, validation). These dialectics provide a useful way to investigate the data; mathematical thinking and acting requires all three, and the degree to which each is present will provide an initial understanding of the students' mathematical learning as it relates to the feedback from the *milieu.* Further interactions (technical and other) were also represented. In order to create a narrative, the coded interactions were placed on a timeline.

Figure 1 below shows part of a timeline and explains how the interactions are represented. On the left are the six categories, and on the right is part of a coded timeline. The section of coded timeline shows when interactions occur in relation to others and the length of each block is directly proportional to the length of time the interaction lasts. This part of the timeline begins with a formulation, which is followed by an action, some computer feedback, another formulation, a technical interaction and one 'other' interaction. It does not include any validation interactions.

Figure 1 Explains the layout and elements of the timeline



GRAPHS WITH AUTOGRAPH

The data discussed below was taken from a series of five lessons for 14 to 15 year olds. From the class of approximately 25 students, the teacher chose a focus group of three girls; Claire, Alice and Charlotte. (These names are pseudonyms).

According to the teacher, the overall aim of the five lessons was for the students to develop an understanding of the relationship between the algebraic and graphical forms of quadratic functions of the form $y=x^2 + bx + c$, where b and c are integers. (The initial interview with the students established that they knew that these graphs would be 'u-shaped' but they had no experience of constructing the graphs by hand.) In some lessons, the software *Autograph*[1] was used to produce graphs of quadratic functions. Some of the lessons took place in the classroom, and some in a computer room. The first of these lessons is analysed below.

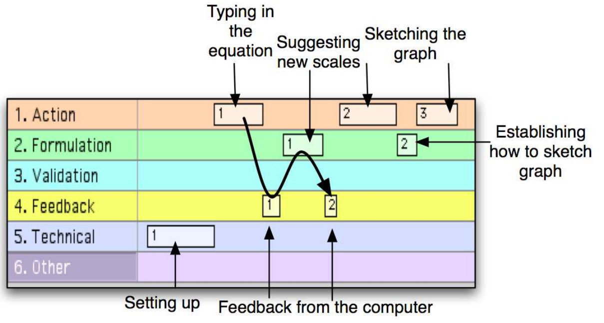
The task for this lesson, set out on a worksheet handed out to the students, was to use *Autograph* to create graphs of quadratic functions and then to sketch the graphs onto a paper-based worksheet. The teacher's stated aim (both to the researcher and the students) for the lesson was for the students to '…notice the intercepts' *[with the axes, but this is implicit]*. (It is argued in the thesis (Joubert, 2007) that the task is unambitious and is unlikely to lead to significant student learning. However, the methodological approach adopted in the study was non-interventionist, and the researcher had no influence on the task chosen by the teacher (perhaps, retrospectively, mistakenly)).

The worksheet consisted of six similar questions, in which students were given an equation, such as $y = x^2 - 6x + 8$ (which the students had already factorised for homework as $y = (x - 2)(x - 4)$).

The students began by turning on the computer and starting *Autograph*. This technical interaction is the first on the timeline (see Figure 2 below). They opened the *Enter Equation* dialogue box and typed in the first function $y = x^2 - 6x + 8$. This action is the next interaction on the timeline. The students went on to complete the question, and Figure 2 below summarises the activity taking place as the students worked on this question.

The timeline shows the various coded interactions. The zigzag (black line) traces the computer activity, and the feedback interactions represent the computer outputs and the students' reactions to these.

Figure 2 Summary of student interactions Q1



This analysis begins to unpick the student interactions with the *milieu*; there are three action dialectics, two formulation dialectics and two feedback dialectics. As will be seen below, the first formulation dialectic is promoted by the computer feedback. It is perhaps unsurprising that there are no dialectics of validation as these were not demanded by the task. However, the dialectics of formulation, and in particular the relationship of these to the computer feedback, are worth exploring.

The first formulation dialectic occurs in response to the computer feedback, an on-screen graph (see Figure 3 below). Alice remarked:

> Can we see it all? Can we change the axes?

Without further discussion, she clicked on the Axes > Edit Axes menu commands. This opened a dialogue box which allowed the students to enter values for *x* and *y* 'max' and 'min' values. Claire suggested the changes needed:

> (*Points to the y minimum*) Change this one to minus two and (*points to the y maximum*) that one to, say, eight. (*Points to the x minimum*) And this one to minus two and, yeah, that will do.

The second graph (see Figure 4) shows the computer feedback after this change. One of the students commented:

> That's better, OK now we can draw that.

Figure 3 First graph, Q1

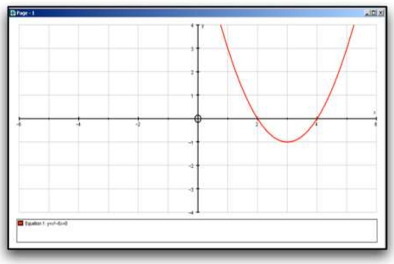
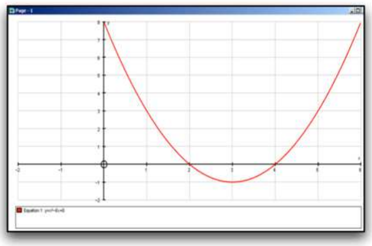


Figure 4 Second graph, Q1



The students then went on to sketch the graph on their worksheets. The second formulation was an unremarkable discussion about how to sketch the graph.

The first point to note about this student activity is that the students did not seem to find any difficulty in using the software. There was no discussion between them about how to create the graph or how to make changes to the scales on the axes. The first computer output was a graph page with the graph of the function drawn on it. However, as Alice's comment indicates, the students wanted to change it. The reason, it seems, was that the *y*-intercept could not be seen. Alice's reaction suggests that the feedback was unexpected, and in response to this unexpected feedback the students changed the graph page.

The students did not discuss the changes, and there is no way of knowing the basis for their decisions. However, it is likely that it was based on the visual appearance of the graph on the page because of Alice's reference to not being able to 'see it all'.

In the remainder of the lesson, the students followed much the same pattern of activity. In general, they tended to type in the equation of the graph, then adjust the graph page by guessing maximum and minimum values (as above) based largely on the appearance of the graph. Only in one case did they sketch the graph on paper before entering the equation into the software (see below).

Two dialectics of validation took place. The first occurred as the students attempted the fourth question. The teacher had encouraged them to sketch the graph first, and then to try it out on the computer but they found a mistake in the worksheet; whereas they had been asked to solve the quadratic equation $x^2 - 8x + \mathbf{15} = 0$ by factorisation, they had then been asked to sketch the graph corresponding to $y = x^2 - 8x - \mathbf{15}$ and in the dialectic of validation the students explained (neither very clearly nor very confidently) why the y-intercept could not be -15. After some further actions and discussion they called the teacher and pointed out the mistake. The teacher asked how they found the mistake. In response one of the students suggested:

> We worked out that it would cut the axes there and there (*pointing at predicted intercepts on the screen)* because those numbers are the same as those (*pointing at the zeroes on the sheet).*

Through her questioning, the teacher had drawn out a justification (second dialectic of validation), by encouraging the students to articulate what they had noticed. However, she did not question **why** the roots correspond to the intercepts, and it seems that both she and the students were concentrating on the visual connections between the algebraic and graphical forms of the function, rather than drawing on the theoretical notions underpinning this connection. Further, it seems that the teacher was relying on the computer to draw out connections by concentrating on noticing rather than explaining.

It could perhaps be expected that the students, having made the connection explicit, might use their understanding to adjust the graph page in further questions, or to sketch the graph before entering it on the computer. Their approach began as follows:

> Alice: OK let's do the next one then.
>
> Claire: What, draw it before?
>
> Alice: No, I'm not doing that – too much hassle.

As the dialogue indicates, it seems that they preferred a trial and improvement approach rather than a more theoretical approach. Their talk indicates that they were deliberately choosing this approach, even though they knew that it was likely that the graph page would need to be changed later (by entering new maximum and minimum values for the visible parts of the page).

In the second dialectic of validation, one of the students justified her choice of scales to the others. Once again, however, she referred to the factorised equation but did not attempt to explain why the factors were related to the roots.

Overall student learning

The brief analysis above suggests that there was some student learning as evidenced by the presence of dialectics of action, formulation and validation. However, it seems that many of the dialectics of formulation were visually based, and were prompted by the computer feedback. The students used a trial and improvement strategy based on the visual appearance of the graph, but there is evidence that on several occasions they made the connections to the equation (initially suggested by the teacher). However, they chose to return to the former approach in later questions. Although they connected the numbers in the equation to the graph but there was no attempt to explain why the connection existed. There is some evidence of mathematical learning, in terms of the theoretical framework used, but the students never explained the connections they found and their learning can be seen as relatively superficial.

CONCLUSIONS

Space has not allowed a detailed analysis of the entire lesson, but the example has served to demonstrate the use of the chosen framework. This brief analysis has, however, suggested that to complete the task the students were not required to engage in dialectics of validation, and it was primarily the teacher's intervention that prompted one of the two dialectics of validation observed. The difficulty, perhaps, lies in a confusion related to the role of the computer, which does the mathematical work of creating the graph, and the question then is, what mathematics will the students do? Further, although the feedback from the computer prompted dialectics of formulation, it appears that it was easier for the students to guess the changes needed and to try these out than to adopt any systematic predictive trial and improvement strategies.

The study overall set out to understand the use of computers in authentic classroom situations; the analysis above has confirmed the importance of understanding the role of the computer. It particular it has emphasised the importance of having a clear idea of the mathematics the computer will do and the mathematics the students will do, and of how feedback can be used so that tasks take advantage of the computer's potential to provoke situations of validation as well as action and formulation.

NOTES

[1] http://www.autograph-math.com/

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