TESTS AND EXAMINATIONS IN A CAS-ENVIRONMENT – THE MEANING OF MENTAL, DIGITAL AND PAPER REPRESENTATIONS

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Digital technologies (DT) face new challenges in working with representations in mathematics classrooms and especially while using them in written tests and examinations. There are unexpected solution strategies of students which are not represented in an adequate – student expected – way, there are – beyond the mathematical competencies – tool competences necessary, and working with handheld technology causes the additional problem that the documentations of solutions have to be fixed on paper or paper representations. Moreover, the student has to manage the interrelationship between her or his mental representations, the digital or tool representations and paper representations. How should the paper documentations look like to give the corrector the chance to track the work of the students? In the following authentic written exam, problems and authentic student solutions in a handheld computer algebra system (CAS) environment are analyzed. Finally criteria for documentations of paper solutions of exam problems are developed.

Keywords: digital technologies, examination, assessment, representation, documentation, computer algebra system (CAS), experimental investigation

REPRESENTATIONS AND DIGITAL TECHNOLOGIES

Digital technologies (DT) open new chances and new ways for the use of representations in mathematics education. DT especially simplify generating representations on the computer screen, they emphasize the experimental working by easily changing parameters, and they give access to the simple use of multiple representations. But there are also difficulties and obstacles while working with digital representations. The speed of the creation of screen representations may overcharge the mental abilities of the user, the interrelationship of different representations has to be developed by the user, and the question of the relationship between the traditional working with paper and pencil and with digital representations arises.

There is a long-standing and on-going debate about the meaning of digital representations in mathematics education (e. g. Guin 2005, Ainsworth 2006, Ladel u. Kortenkamp 2009). Opportunities and problems are already discussed in the 1st ICMI study about DT: The Influence of Computers and Informatics on Mathematics and its Teaching (Churchhouse 1986), it is an essential aspect of many ICMI activities in the last 25 years (Laborde & Sträßer 2010), and it is still of essential meaning in the latest 17th ICMI study: Mathematics Education and Technology – Rethinking the Terrain (Hoyles a. Lagrange 2010).

There is not much known about the interrelationship between these different kinds of representation while working with digital technologies in a test or examination environment. There is a huge section (p. 81-284) “Learning and assessing mathematics with and through digital technologies” in the 17th ICMI Study, but it is about theoretical frameworks about learning with DT, changes in mathematical knowledge and practices resulting from access to DT, learning trajectories, automatic assessment and social learning, and it does not give answers to questions concerning the influence of DT on tests and examinations in the classroom. Especially, the use of DT in assessments and examinations is not explained in the frame of empirical investigations or in analyzing students’ work in examinations.

THE LONG STANDING SYMBOLIC CALCULATOR M^3-PROJECT

A long-term project (2005–2012) was started to test the use of symbolic calculators (SC[1]) in Bavarian grammar schools (“Gymnasien”) in Germany in grades 10 to 12 (the M^3-project[2]). The students are allowed to use SC in class, for their homework and in all tests and examinations (see Weigand & Bichler 2010a, 2010b).

The results of the evaluation of some tests in these classes showed that students in the project classes still have – also after one year of SC usage – difficulties in using SC and *(problem-) adequate representations* especially as well as the *documentation of the solution* with paper and pencil. In a questionnaire at the end of the school year, nearly 40 % of the students in the project classes reported to have difficulties in using the SC, many students assigned these to “technical difficulties”. But from interviews with and video studies from students, we learned that the real reasons for these difficulties are quite often on the mathematical and not the technical side. Students don’t know how to use representations and how to get some information out of them.

In May 2010 the first final baccalaureate examination was given to the students at the end of school time in grade 12. In Bavaria the baccalaureate is a state-wide examination with the same problems for all students, given by the ministry of education.

EXAMPLES FROM THE M^3-PROJECT

In the following we concentrate on two main aspects of the above mentioned difficulties: how to choose the adequate representation in a problem solving process and how to document the solution of a problem. We will explain these aspects and difficulties in the frame of test and examination problems and authentic student solutions of these problems.

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[1] We used the TI Voyage 200, TI-Nspire and Casio ClassPad

[2] M^3: **M**odel project new **M**edia in **M**athematics classrooms

Adequate representations

A representation may be considered as *adequate*, if it represents situations or helps to solve problems the way one wants. In fact, representations always have to be considered in connection with possible and appropriate operations. The following example is taken from a test written in 10th grade.

Example 1: Given are f and g with f(x) = sin(x) + 1 and g(x) = 2^x. How many intersection points have the graphs

a) in the interval x ∈ [10; -10] and

b) for x ∈ ℝ? Give reasons?

Solving this equation on the symbolic level with the SC gives the result (Fig. 1):

$$\text{solve}\left(\sin(x)+1=2^{x},x\right)$$
$$x=^{-}70.6858 \text{ or } x=^{-}70.6858 \text{ or } x=^{-}2.23478 \text{ or } x=0. \text{ or } x=.749645$$

Fig. 1. A warning sign appears in the display: “Some more solutions may exist”.

It is quite difficult – for students (nearly) impossible – to interpret this display’s numeric solution. Changing to the graphic screen and zooming into interesting sections is a good strategy. This gives the following graphs:

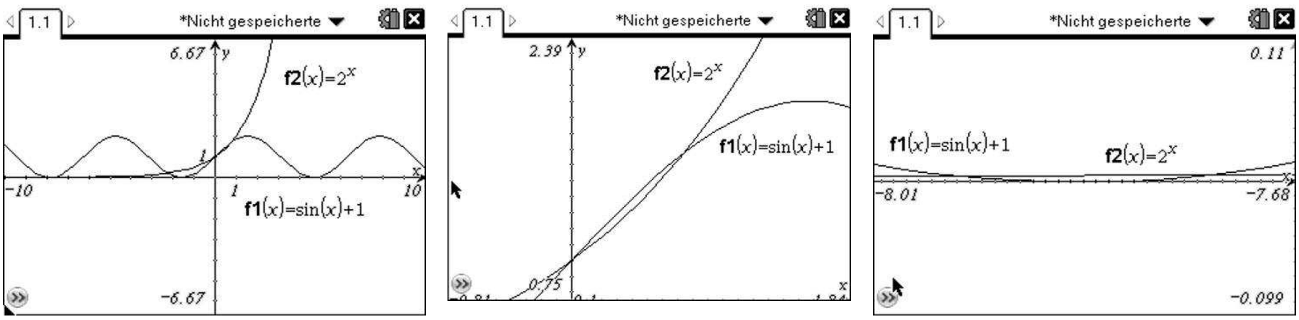

Fig. 2. Screen shot of the functions with *f(x) = sin(x) + 1* and *g(x) = 2^x*.

To indicate the “interesting sections” and to interpret the graphs – for the area x < 0 – require advanced basic knowledge of the properties of the sine and the exponential function. The solution – an infinite number of intersection points – cannot be obtained from the calculator screens, it has to arise from the basic knowledge concerning the implied *mental representation* of the functions. 30 % of the students are able to solve the problem 1 a), but less than 5 % are able to solve the problem 1 b) with the general domain ID = ℝ.

Unexpected digital representations

The following problem is from the final baccalaureate examination in Bavaria in May 2012. The text is given here only in a short form.

Example 2: A ceramic art in the Casa Batlló (see Fig. 3) has roughly the shape like a parabola.

a) *Give a model of the shape of the upper prim of the art work by using*
   $q(x) = ax^4 + bx^3 + cx^2 + dx + e$
   (For control: $q(x) = -0.11x^4 - 0.81x^2 + 5$*)*

b) …

c) *The line g is parallel to the x-axis and divides the work of art into two sections. The area of the above segment should be 71.5 % of the whole area.*

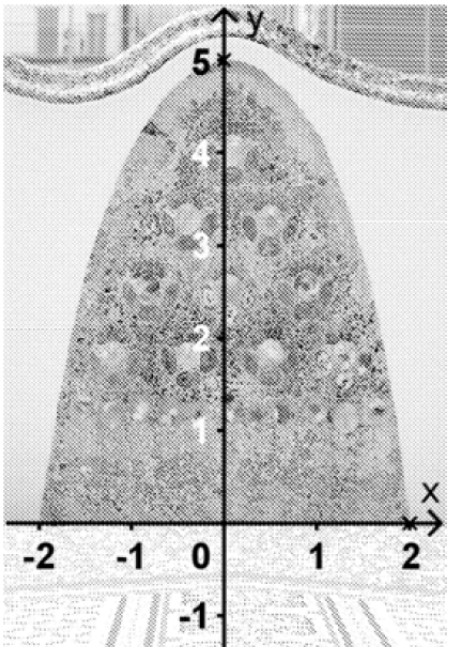

Fig. 3. The art work of Gaudi

We concentrate only on problem c). The situation shows the following screenshot (Fig. 4). You first have to calculate the integral of q from – 2 to 2 (Fig. 5).

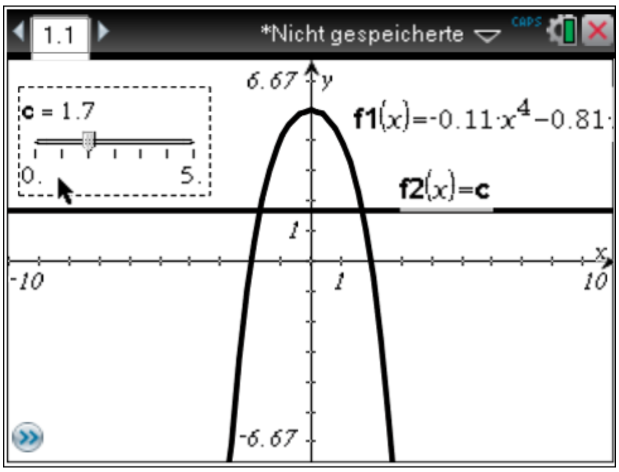

Fig. 4. The graph G_q and the parallel line

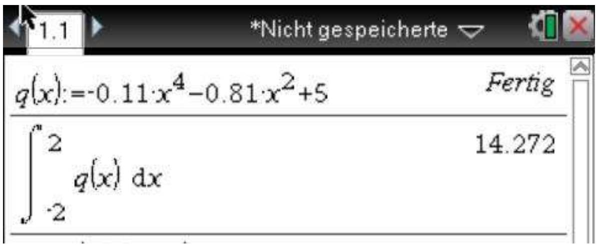

Fig. 5. Calculation of the integral

If you solve the equation q(x) = c you get – with the handheld TI-Nspire – the following result (Fig. 6).

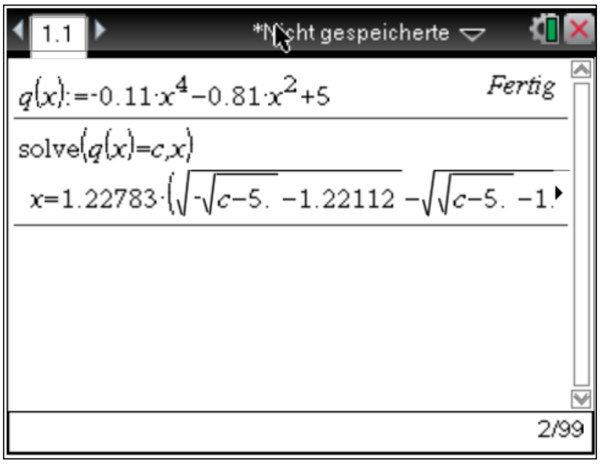

Fig. 6. Solution with the TI-Nspire

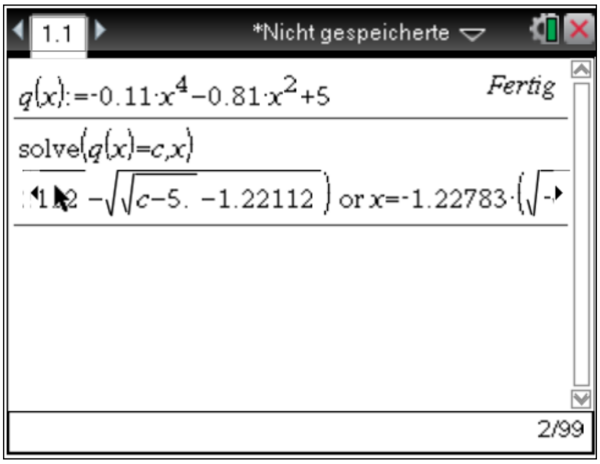

Fig. 7. Scrolling right

The small triangle at the right side of the last line of the screenshot shows that the formula is not finished at the end of the screen. Scrolling to the right shows a surprising long line (Fig. 7). It is impossible – not only for students – to interpret this result. But also with a laptop representation (which was not available for the students in the exam), the result stays confusing (Fig. 8).

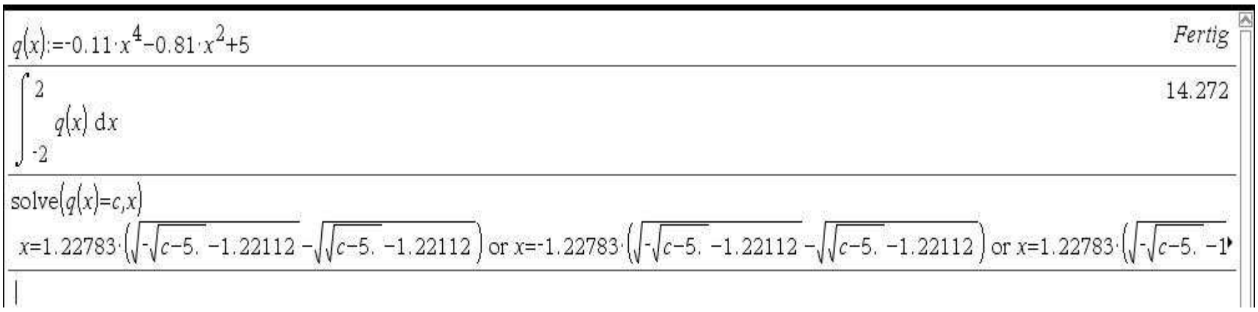

Fig. 8. Solution with the TI-Nspire Notebook Version

Moreover, you see that the expression $\sqrt{c-5}$ in all solutions shows that there seems to be no real solution for c < 5, which is obviously wrong, because there exist two real solutions for $0 < c \leq 5$ (Fig. 4).

This example shows the problems which occur if the problem poser had another solution in mind and did not think of different solution strategies.

The documentation of solutions

Quite often – especially in written tests and examinations – the solution of a problem has to be documented on paper. This opens the question concerning the *adequate documentation form* of the solution on paper.

Example 3: Given is the function f with $f(x) = (x - 2)^2 + 3$. Determine the equation of the tangent in the point P(1 / 4).

The following examples give some students' solutions and show different documentations. The underlinings show the use of the handheld device of the student during the problem solving process.

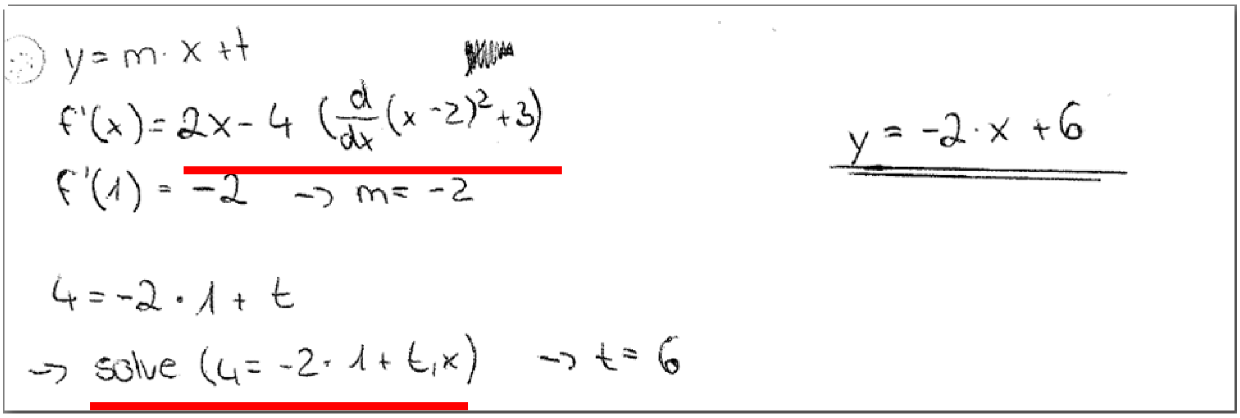

Fig. 9a. A student solution quite similar to a paper and pencil solution

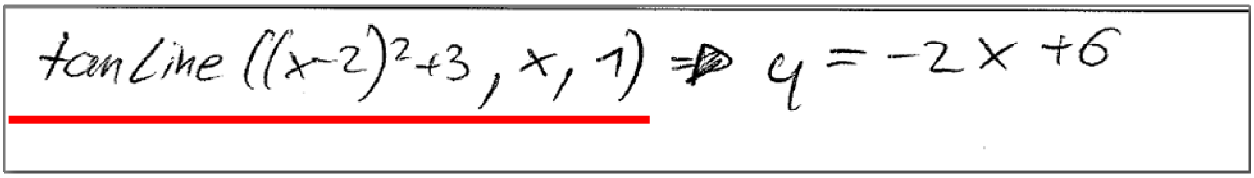

Fig. 9b. A student solution using a handheld command

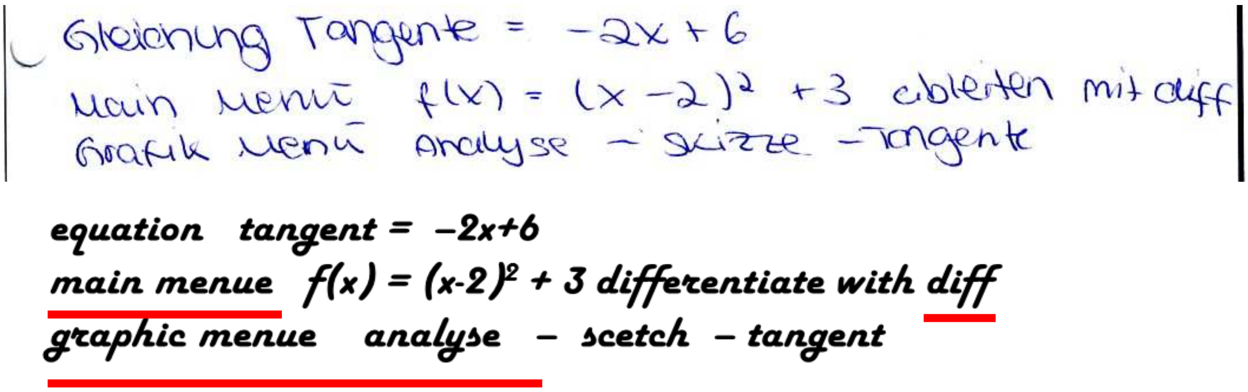

Fig. 9c. A student solution and the English translation

You might be content with the solution 9a, solution 9b might be accepted in a DT-environment, but it needs a special knowledge about DT-commands. But – for sure – you will not be content with solution 9c!

Example 4. Give a sketch of some graphs of f_a *with* $f_a(x) = a \cdot x^3 - 3x + 1,\ a \in \mathbb{R}$. There are still some reasons why it might be useful or helpful to do a sketch of a graph (also) by hand, especially in relation with heuristic problem solving strategies. The question about the function and the expected accuracy of hand sketches arises. Fig. 10a and b show a screenshot and a student's hand draft of graphs of a family of functions. Even for heuristic reasons the hand draft does not fit to an expected accuracy and for sure it is not accepted as a solution of example 4.

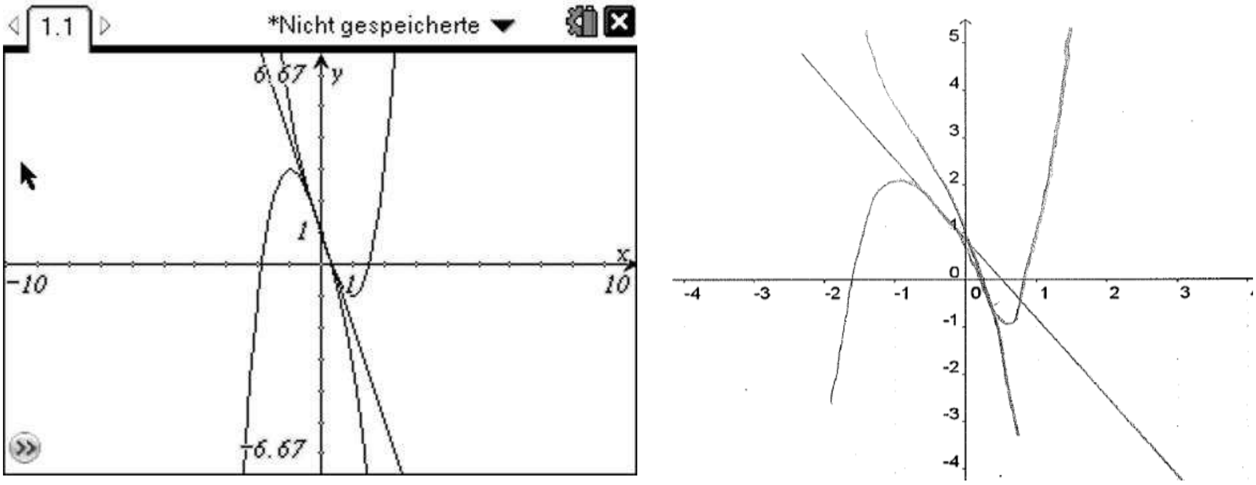

Fig. 10a and b. Digital and hand graphs of f_a with $f_a(x) = a \cdot x^3 - 3x + 1$ for a = -1, 0, 1

EXAMPLES FROM THE FINAL BACCALAUREATE EXAMINATION

The following example is also from the final written baccalaureate examination in Bavaria in May 2012.

E. 5: Given is the family of the in $\mathbb{R}$ defined functions f_a: $x \rightarrow \sin(ax)$ with $a \in \mathbb{R}\setminus\{0\}$.

a) Give two values of a, such that all zeros of f are integers.

b) Now a = 2. Calculate $\int_1^b f_2(x)\,dx$ *; b is the smallest zero of f_2 which is bigger than 1. Give a value $c \in \mathbb{R}$ with c > b, that* $\int_1^c f_2(x)\,dx = \int_1^b f_2(x)\,dx$. *Give reasons for your answer using the graph of f_2.*

With the use of DT the solution of example 5b might look as follows:

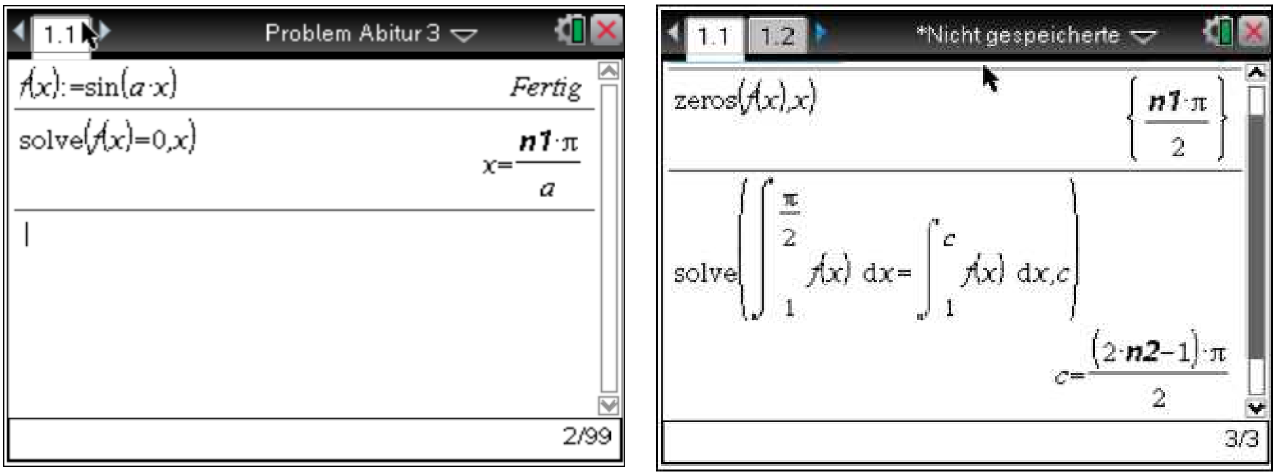

Fig. 11a and b. The DT-solution of example 5 a) and b)

These solutions (Fig 11a and b) are prototypic for DT-solutions. The student has to find the problem solving idea and the starting equation, the calculator takes over the former hand calculations, and finally the student has to interpret the screen notations.

This example especially shows that it is (nearly) impossible to do this interpretation without a basic mathematical content knowledge about the given functions and concepts, e. g. the integral concept.

Working with DT it is always a good strategy to choose different representations of a problem. Fig. 12a and b show graphical representations of the problem which allow a solution of the problem without calculations.

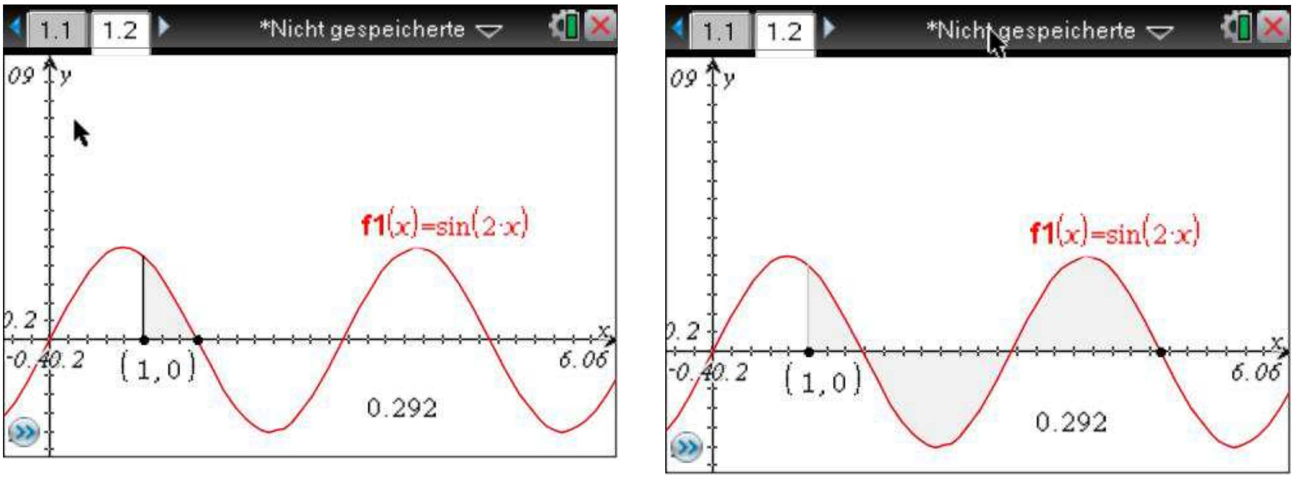

Fig. 12a and b. A graphical representation of the solution of example 5 a) and b)

Fig. 13 shows a student solution of example 5b) with a well-written reasoning concerning the solution. The problem for the teacher and the corrector of this examination is that the solution does not show whether and how the calculator was used.

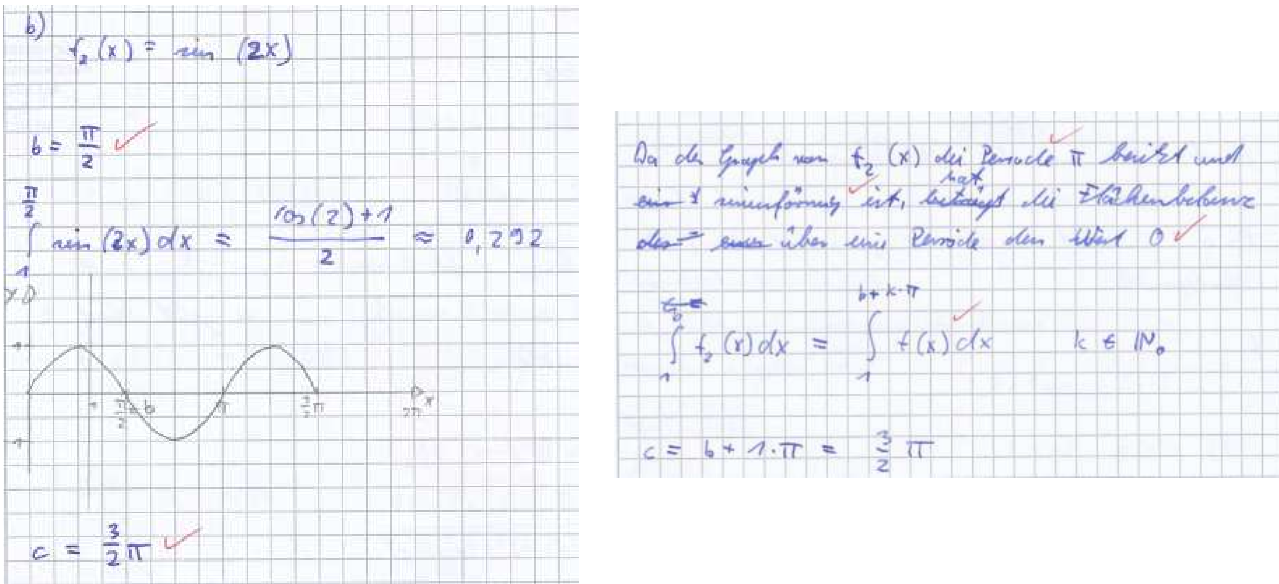

Fig. 13. A – correct – student solution of example 5b)

CRITERIA FOR DOCUMENTATIONS OF SOLUTIONS

Written examinations of students with DT ask for clear instructions for the documentation of written solutions. But there are no algorithmic rules or norms how to document a solution on paper. This opens the question concerning the *adequate documentation form* of the solution on paper.

In our project we started to develop *criteria* for (non-)correct, (non-)accepted documentations of solutions, e. g.

- It is not enough to only write down, what‘s on the screen!
- The solution has to be understandable „for others“, and it has to be seen when and where the SC was used.
- The solution describes the mathematical activities, it is not only a description in a special „calculator language“.
- The meaning of “keywords” (operators) in the problem definition has to be well-known to the student, e. g. “show”, “explain”, “determine”, “prove”, …

These criteria have to be discussed, evaluated and refined the next years.

FINAL REMARKS

This analysis shows the problem and difficulty of posing adequate test problems and the problem of the adequate students’ paper documentation of the solution of the problem. The ability or the competencies *to use SCs* adequately requires technical knowledge about the handling of SCs. Moreover, the knowledge of when to use which features and for which problems might be helpful.

The use of the adequate representation depends – of course – on the problem and the expected level of accuracy or strength. Furthermore, the problem solving process requires the knowledge about the *relationship* between *mathematical objects* or *concepts* and their *mental representations* and finally between the digital representation and the representations used in the documentation (paper and pencil representations). Despite the importance of different interactive multiple (digital) representations, the most important representation, which had to be developed, are the *mental representations*.

Concerning future developments there are some important research questions concerning the adequate use of representations in a digital examination environment:

- How are mental representations influenced while working with digital representations and vice versa?
- How should examination problems be posed to allow a greater variety of problem solving strategies while using DT?

- Are there general criteria for the documentation of problem solutions which cover a greater range of problems?
- Which documentation criteria are helpful for students?

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