PERFORMANCE ON RATIO IN REALISTIC DISCOUNT TASKS

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In this paper we present a qualitative analytical tool to support the design of a protocol for teaching ratio and proportion tasks, structured around metacognitive principles taken from Socratic maieutics.

In addition, we present an example of its implementation for categorizing responses that a group of pre-service mathematics teachers gave in a ratio discount task. The data presented in this paper is part of an empirical study on students' response patterns, which is considered essential for the design of this maieutic proposal.

Key words: Ratio, Proportion, Discount, Comparison, Metacognition. Maieutic

INTRODUCTION AND THE RESEARCH OBJECTIVE

This study is part of a broader project – theoretical, empirical, and educational – which has been structured on the design, elaboration and implementation of a teaching proposal for ratio and proportion tasks, aimed at pre-service teachers.

This proposal follows the idea that "effective learners recognize the importance of reflecting on their thinking and learning from their mistakes", as pointed out by research and curricular orientation such as the NCTM (2000, p.20) in the United States. It highlights the role of metacognition in the mathematics classroom, as indicated by Lester (1985) and Schoenfeld (1985).

Teaching proposals based on metacognitive practices rest on the assumption "that metacognition demands to be taught explicitly" (Desoete, 2007 p.709). Despite this, "teachers still pay little attention to explicit metacognition teaching" (Ibid, p.709).

In this regard, Socratic maieutics (conceived by Socrates and put forward in Plato's dialogue *Meno*), is considered to be a suitable pedagogical method as it promotes and specifies metacognitive processes to "foster learning in students from self-recognition of their ignorance" (Rigo, 2011, p.523).

In the Socratic Method three key moments are identified: Construction, De-construction, and Re-construction (Rigo & Gómez, 2012).

In the construction moment, the teacher poses a task that he/she knows students will answer confidently but in an incorrect or limited manner. On being asked to justify the answer, students engage in metacognitive reflection, which takes into account individual variables (confidence in the answer), task variables (knowledge of the mathematical notions involved and the degree of difficulty of such a task), and strategy variables (which strategies were available to solve the task and which was chosen and why). These reflections are necessary to prepare students for the transition towards the moment of de-construction.

Next, the teacher leads the students to confront cognitive and metacognitive conflicts. Cognitive conflicts arise when students are confronted with the contradictions that emerge from wrong answers. Metacognitive conflicts arise when students reflect and become aware of the limitations of their answer and ideas about the subject matter. This is the so-called moment of de-construction.

Lastly, at the moment of re-construction, the teacher guides the students so that they can produce a new answer that will enable them to understand what has, up to that moment, been unknown in terms of the proposed task.

For this maieutic process to be effective and have the expected cognitive and metacognitive impact, the chosen task must be suitable for maieutic purposes. Furthermore, it is essential to know possible response patterns beforehand and to have determined unresolved difficulties.

These response patterns will serve as a guide for maieutic teaching sequences that will allow us to design the metacognitive-maieutic questions introduced later during the de-construction stage, and also plan the cognitive conflicts which could eventually be used as learning opportunities.

The data presented in this paper refer to a categorization of response patterns given by a group of pre-service mathematics teachers in a discount comparison task. This is considered essential, as mentioned earlier, for the design of our maieutic proposal.

METHODOLOGY

To categorize the responses we used a qualitative analytical tool. It consists of an interpretation scheme that is based on common features to group the different responses given by pre-service teachers when try to solve the tasks, in the maieutical construction moment.

The mode of delivery the task is a worksheet (paper and pencil format) administered during a regular one-hour class session, to 314 third-year students undertaking a primary education degree at the University of Valencia (Spain). These students had already completed an annual course in mathematics and at the time of the study were taking another course in mathematics teaching.

Two criteria were used to select questionnaire tasks: the first corresponds to representative or prototypical examples in each area identified in Solomon's (1987) conceptual schema (arithmetic proportionality, scales, Thales and slopes). This schema covers the network of ratios and phenomena that are organized around ratio and proportion. The second involves tasks which are considered to be maieutic, i.e., tasks that meet the following requisites:

- Rich in interpretations and meanings.
- Rich in math concepts that a future teacher should know.
- Can be solved in different ways.
- A familiar task.

- The answer is likely to create a high degree of confidence.
- Apparently simple, but can create difficulties.

Due to limited space, in this paper we will only discuss the results of numerical proportionality task called "the discount comparison task".

The discount comparison task

Students are presented with three typical advertisements and asked which discount is better (Figure 1).

Figure 1. The discount comparison task

The three advertisements are phrased differently. The first advertisement shows a percentage, “70% off second item”. The second one shows “3x2” and the third “Second item half price”.

The last two discounts are applied to a particular product: the “3x2” advertisement offers three bottles of “Rioja” wine for the price of two; the other advertisement offers the second bottle of “Extremadura” wine at half price.

For added convenience, offers that apply to a product include unit values: in 3x2, the price of each bottle is given before and after the discount: €5.58 and €3.72; and in the second bottle half price, the price of the bottle before the discount is given, €9.74 and the price of the second bottle after the discount is given, €4.87, thus avoiding arithmetic calculations.

The discount as a ratio

When ratio refers to a way of comparing two quantities of the same type, as for example, in “there are 3 boys for every 4 girls”, or “3 out of every 4 people smoke”, the comparison is, in the first situation, part-part since the quantities are separated, and one is not contained in the other; and in the second situation the comparison is part-whole.

What is characteristic in these situations is:

- Relational number. The fraction 3/4 symbolizes a set of ordered pairs (3, 4); (6, 8); (9, 12), …, which express an invariant relationship between the numerator and denominator that quantifies the ratio regardless of the total quantity.
- Co-variation. The underlying idea that any change in the numerator will produce a change in the denominator because the ratio must be maintained.
- Invariability. It is not necessary to know the "whole" because the ratio does not change the value when the total quantity of the whole is changed.
- There is no partition or fracture, and there is not necessarily to know a natural unit or "whole", as in other aspects of the meaning of fractions.

The discount is "a deduction from the usual cost of something" (Word reference). This quantity can be given indirectly as, e.g. "3x2", "70% off", or "half price".

"3x2" is a ratio since it symbolizes a relational number, the set of pairs (3,2), (6,4), (9,6), …, co-variation, the ratio does not change when the quantity of the whole, and invariability, a change in one element of the pair, produces a change in the other.

70% is also a ratio since it meets the same criteria: (7, 10) (14, 20), ….The same occurs with "half price" which symbolizes pairs (1, 2), (2, 4), …

The difficulty in the discount comparison task as a ratio lies in the fact that the data refer to ratios that are not expressed in a comparable way, so a conversion of the data must be made to make it possible.

The comparison can be carried out in several ways, e.g. "3x2" can be expressed as percent: "you get 1/3=33.3…% off each item for every three items purchased", and "70% off the second item" can be expressed as "you get 70%/2=35% off each item for every two items purchased". Likewise, "Second item half price", can be expressed as "you get 50%/2=25% off each item if you purchase two items".

One way of answering the question is by stating "Buying 6 units of each product takes advantage of the three types of discounts and the best deal is the one offering 35%, which is 70% off on the second item". However, there are other possible answers.

Criteria to analyse the data

To describe student output, we examined the critical components of task employed by students: numerical relationships, unitizing, and unit rate strategy.

The result of these provides us a scheme with categories, which are subdivided into subcategories, and these in classes and subclasses, with propose to have a more detailed description of the students' response patterns.

The nature of the numerical relationship that exists in the three advertisements is the first critical component: relational numbers, co-variation and invariability (i.e. ratios).

The second critical component is unit reference to the ratios used to compare.

In “3x2” and “70% of second” or “Second item half price”, the ratios have different unit reference, so is required a process to construct a comparable unit reference (reference-transforming units) and then to interpret the situation in terms of that unit (see *unitizing* process in Lamon, 1996).

But, in “70% of second” and “Second item half price” unit reference is the same, so a numerical answer is not required, however the ratio 70% and 50% have to be compared, as in *numerical comparison problems* (Cramer & Post, 1986, in the context of proportional reasoning).

The third critical component is the *unit rate strategy*. “As the name implies this is a *how many for one?* strategy” (Ibid, p.406). For example, 70% of second item is $\frac{70}{2}$%=35% of discount per item. Second item half price is $\frac{50}{2}$%=25% of discount per item, and “3x2” is 33.3…% of discount per item.

The schema

Consequently, the first criterion considered to group the responses is, if students perceive the discount as a ratio or not. This determines two main categories: Discount as a ratio and Discount depends on specific elements of the offer.

To distinguish subcategories in the first category, the criterion is to determine if the students’ approach is to compare elements that are comparable or elements that are not.

For distinguishing classes, the criterion is if students’ strategy focuses on a unit value. Subclasses are obtained according to the quantities that compare: percentage or discount per item, and cost or the number of items purchased.

In the second category, we include responses by the students who perceive the discount as a feature depending on the item to be purchased.

Subcategories are established by distinguishing between qualitative and quantitative approach. The first subcategory gathers those responses that focus on aspects such as quality of the product, size of the bottle, and so on. The second subcategory includes responses that use particular numerical data of the discounts.

In the quantitative subcategory, as a criterion to distinguish between classes, we observe whether the students focus on the operation (with the prices in the ads) or the number (how many bottles I am given).

Figure 2 graphically organizes these categories, subcategories, classes, and subclasses.

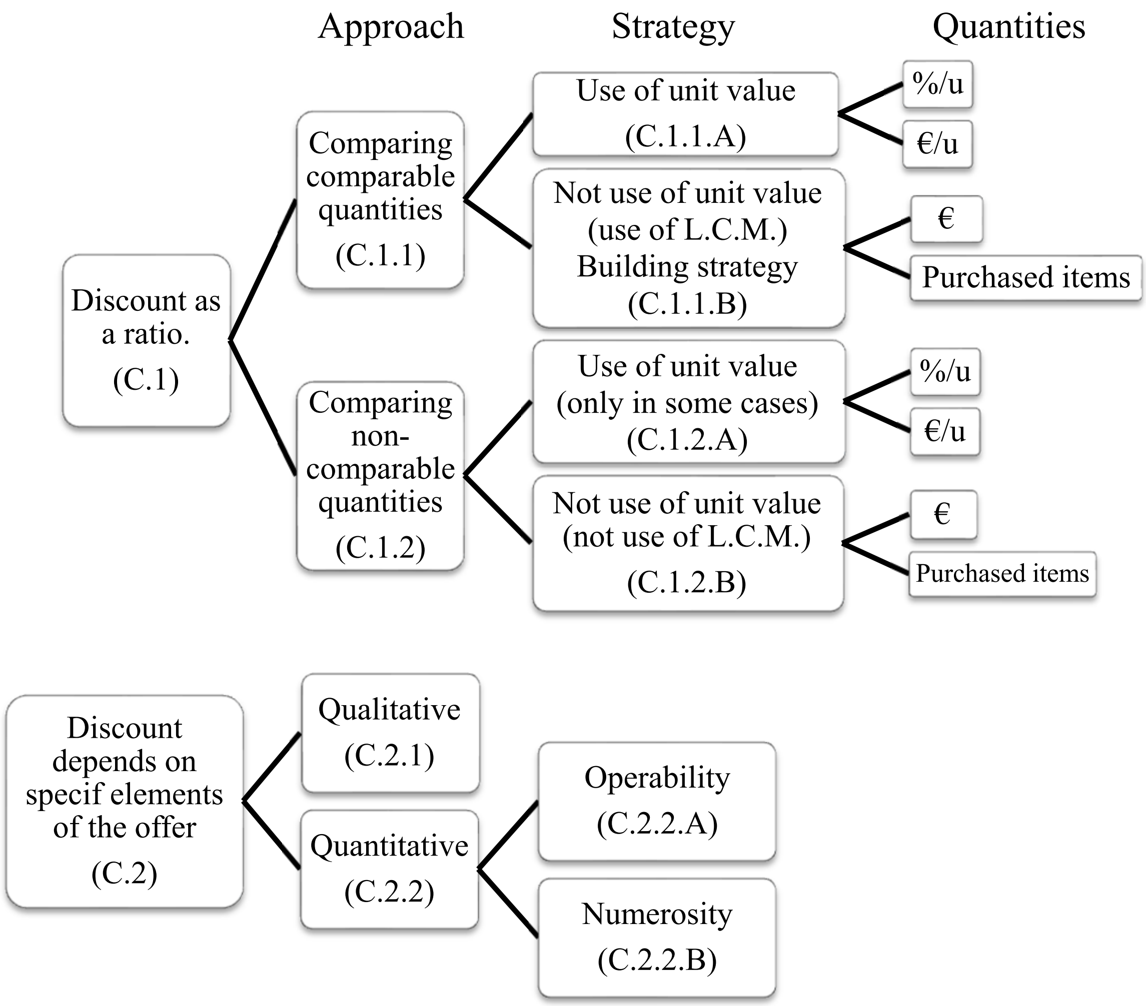

Figure 2. Categories, subcategories and classes

EXAMPLES

A description of each response pattern follows.

C.1.1.A. Discount as a ratio. Compare comparable quantities. Use of unit value.

In this class there are two response patterns: the percentage of discount is calculated for each item; and the unit price for each offer is calculated based on the average price.

Example 1. In the first offer, Maria uses the rule of three to calculate the percentage paid for each item. Her calculations are: 100/5.58=x/3.72; x=100x3.72/5.58=66.66%. From this, she obtains the percentage of discount per item by subtracting 100-66≈33%. In the second offer, the student divides 50%÷2 =25%. In the third offer the student divides 70%÷2=35%. Finally the student states that "the best deal is 70% off the second item.

Example 2. Juan says: “the item costs €5. Applying 3x2→3 items €10, 10÷3=€3.3 for each item. Applying second item half price: 1st item is €5, 2nd item is €2.5, adds up to €7.5, I divide 7.5÷2=€3.75 for each item. Applying the discount to 70% off second item, the 1st item is €5, 2nd item $\frac{70}{100}$x5 = €3.5; 5-3.5=1.5 2nd item". Next, he writes: 5+1.5=6.5, he divides 6.5÷2=3.25, and he states: "The best deal is 70% off second item".

The students have compared comparable quantities: one student has reduced the percentage discount to unit prices in each offer; the other student has calculated the unit by assigning the price of €5 to the items in each offer.

C.1.1.B. Discount as a ratio. Comparing comparable quantities. No use of unit value.

In this class there are also two response patterns: the number of bottles that would have to be bought is determined (example 3); the costs are calculated by assigning the same price to all the bottles (example 4).

Example 3. Paco claims that "at first glance we wouldn't be able to make comparisons so we find the least common multiple of the items we purchase, which in this case LCM (2,3)=6.

	Items we get	Purchased items
3x2	6	4
Second item half price	6	4.5
70% off second item	6	3.9

Now we can make comparisons. In this table we can see that the best deal is the third one (70% off second item)."

Example 4. José states the following: "If a bottle costs €10: 3x2 means that you get 3 and pay for 2, i.e. 3 bottles for €20, 6 bottles for €40, 9 bottles for €60. 70% off second item: 70% of €10 is €7 →second bottle €3, so 2 bottles €13, 4 bottles €26, 6 bottles €39, 8 bottles €52; 50% off second item (half price), 50% of €10 is €5→second bottle €5, so 2 bottles €15, 4 bottles €30, 6 bottles €45, 8 bottles €60. The best deal is 70 % off second item".

These students have compared quantities that are comparable, but instead of finding a unit value as in the previous subcategory, they have calculated the least common multiple of the number of bottles offered (2 and 3) and they have taken this into account to calculate the number of purchased items or the cost assuming that each bottle is priced at €10. This is a *building up strategy* (see Hart, 1981).

C.1.2.A. Discount as a ratio. Comparing comparable quantities. Use of unit value.

In this class there are two response patterns that are similar to those in C.1.1.A, but a unit value is not used in all the offers since the students end up comparing the percentage or the unit price in the 3x2 offer with the percentage or cost of the discounted item (the second item) in the other two cases.

Example 5. Jesús says "the highest discount percentage is the best". First he calculates the price of three bottles of Rioja without a discount: "5.58x3 =16.74; next he calculates the price with a discount: "3.72x3=11.16". He uses the difference between these two prices (16.7-11.16=5.58) to calculate the percentage of the unit price by using the rule of three: 5.58x100÷16.74 =33.33%". However, in the second and third offer he directly applies the discount that only refers to the second item: "70% and 50%"

Example 6. Susana writes the following: "Price: 3€. Discounts: 70% off → 3x70÷100=2.1 which is 70% that must be subtracted from the item, 3- 2.1= €0.9. R: €0.9 applying 70%. 3x2→ the product costs €3 and we would purchase 2, 3x2=6, as we get 3 products we divide: 6÷3=€2. R: €2 applying 3x2. 50%→ 3x50÷100 =1.5 which is 50%; 3-1.5= €1.5. R: €1.5 applying 70% off. As we can see the best deal is 70% off."

Here students do not calculate the discount per item purchased in the second and third offer and they directly deduct 50% and 70%, which is only applied to the second item to compare it with the discount per unit purchased that has been calculated for the first offer.

C.1.2.B. Discount as a ratio. Comparing non-comparable quantities. No use of unit value.

In this class, there are also two response patterns similar to those in C.1.1.B, but when calculating the students apply the discount to the same number of bottles, but they do not use a common multiple, thus making it meaningless to compare these items.

Example 7. Patricia points out for example "5 items: 70% off→you pay for 3.4 (mistake, is 3.6) items, 3x2→ you pay for 5 items, second item half price→ you pay for 4 items. The best deal is 70% off".

Example 8. Marta writes "each item costs the same, for example €10, and in the 1st I would buy three items for €20 (€10 savings); in the 2nd three products would cost €25 (5€ savings); and in the 3rd three items would cost €27 (€3 savings); therefore the best deal is the 1st".

C.2.1. Discount depends on specific elements of the offer. Qualitative.

This subcategory includes those responses that are not a consequence of numerical calculations but have been worked out based on the features of the products that appear in the advertisements.

Example 9. Ana says "I would say that it depends on the product and the quality as to whether it will be cheaper or more expensive. In addition, it is important to see the amount in each bottle. With this information, relations can be established."

For this student, the concept of discount is linked to qualitative elements of the wine.

C.2.2.A. Discount depends on specific elements of the offer. Quantitative. Operability.

In this class the responses that follow a quantitative strategy are grouped together. The students perform arithmetic operations with the intention of providing a result, without making the required transformations to obtain comparable elements.

Example 10. Laura carries out the following operations: 3x2, 3.72x3=11.16→3 bottles. In the case of the second item half price: 4.87+9.74=14.61→2 bottles. She concludes by saying:

"1st =3 bottles=€11.16 ; 2nd =2 bottles=€14.61; Here you can see the cheapest option"

This student's solution is conditioned by the prices and number of bottles that appear in each advertisement. Thus, she ends up comparing the total cost of buying three bottles of the first wine (which has a price) with the cost of buying two bottles of the other wine (which have a different price). She does not use the 70 %'s option because there are no bottles to apply it.

C.2.2.B. Discount depends on specific elements of the offer. Quantitative. Numerosity.

This class groups together those responses in which the quantity of items purchased gives the solution to the proposed task above the other aspects (such as those that are related with the cost).

Example 11. Pablo writes "With 3x2 you buy two bottles and get one free, without having to pay anything. 9.74x2=€19.48 for three bottles. With 50% you pay €9.74 for one and €4.87 for the other, so you buy two and you do not get one free. With 70% you buy two bottles, one for €9.74 and the other for €2.92 so you pay for two bottles. Therefore, the best deal is 3x2 because you buy two and get one free."

For this student the best discount depends only on the number of items you get for free when you buy, in the case of 3x2, three bottles and in the other two cases, two bottles.

FINAL REMARKS

In this paper we categorize data analysis related to a task that involves ratio and proportion notions. The results demonstrate that the selected "discount task" is a maieutic task.

Through the response patterns classified, we should continue with the teaching maieutical protocol, working with pre-service teachers in normal sessions. Now, in a same session, we can propose the task and implement the metacognitive maieutical questions because (although unknown the student's specific answers) we know in advanced their way of thinking and their possible responses pattern.

In this session the maieutical questions aren´t improvised, because they correlated the response patterns known. With the questions: How did you solve it, what did you base it? , what do you know or unknown about the subject? , we expect that students are confronted with their intuitive notions of the discount and ratio became aware of the limitations of their ideas. In addition, with the questions: in what unit reference have you focused? , or what do you think of your strategy, what do you base it on? , we expect that students explain their reasoning and to listen and to make sense of others' solutions, paying attention to the role of unit reference, and unit rate setrategy.

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