LEVELS OF OBJECTIFICATION IN STUDENTS' STRATEGIES

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The aim of this paper is to identify students' strategies while solving tasks which involve the expansion of fractions to a common denominator. In this case study we follow two groups of 11 year old students and their use of the artefact multilink cubes in the solution process. The analysis of the students' strategies is based upon a semiotic-cultural framework. Five different types of strategies are reported: trial and error, factual, contextual, embodied-symbolic and symbolic. The naming of these strategy types is inspired by Luis Radford.

Key words: Artefact, common denominator, strategy and levels of objectification.

INTRODUCTION

A lot of research has been carried out involving embodied cognition and the multimodal paradigm (Arzarello & Robutti, 2008; Gallese & Lakoff, 2005). Such studies also encompass gestures and the use of various artefacts. Within the semiotic-cultural framework, learning has been formulated in terms of objectification (LaCroix, 2012; Radford, 2008), but to our knowledge this theory has not yet been applied to physical artefacts in the learning of fractions. In this paper we follow two groups of 6^{th} grade students who use the physical artefact multilink cubes to solve tasks that involve expanding fractions to a common denominator. We focus on how students use multilink cubes and mathematical signs equipped with a cultural meaning to express and communicate their thinking in social interaction. Radford (2010b) has described mathematical thinking in the following way: "[…] thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts (p. XXXVI)." A main point here is that mathematical thinking entails the use of resources located outside of the brain, and that such resources play an important role in mathematical activity. Radford's theory of objectification (2006) will be a theoretical foundation for our study:

> The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning "to throw something in the way, to throw before". The suffix – *tification* comes from the verb *facere* meaning "to do" or "to make", so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent − e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property (p. 6).

An important point in this theory is that learning is closely connected to actions aimed at noticing different aspects of the mathematical object at hand. We use Radford's definition of a mathematical object (2008): "[…] mathematical objects *are fixed patterns of reflexive human activity incrusted in the everchanging world of* *social practice mediated by artifacts* (p. 222).” This definition emphasises that mathematical objects are patterns of activity closely linked with the use of artefacts. The mathematical object we study is “the expansion of two fractions to a common denominator”. This procedure is a “*fixed pattern of reflexive human activity”*, so it fits well with Radford’s definition of a mathematical object. The theory of objectification is used as an analysing tool in order to identify deeper levels of objectification in the students’ strategies for expanding two fractions to a common denominator. The research questions that guided our work were: “Which strategies do the students employ as they expand two fractions to a common denominator, and what aspects of the expansion process are at the centre of the students’ attention in these strategies?” In relation to students’ generalisation of number patterns, Radford has described the factual, contextual and symbolic layer of generality (2006, 2010a, 2010b). Radford also refers to these levels as levels of objectification. In this paper these levels are generalised and applied to the students’ strategies for expanding fractions.

METHOD

The case study was carried out in a 6th grade classroom in the autumn of 2011 in Norway. In cooperation with the teacher we selected two groups of three students who were medium to high achievers. We did not choose low achievers because the students would encounter the expansion of two fractions to a common denominator one year before what is normal in Norwegian schools. Every group had 13 sessions of 45 minutes with one of the researches, and all of these sessions were videotaped. The groups were given a problem to solve, and afterwards they were asked to explain how they reasoned as they were working with the task.

The students used multilink cubes to build rectangular “chocolate bars” to depict different fractions. The brown cubes illustrated brown chocolate, and they corresponded to the numerator. The white cubes illustrated white chocolate, and the total number of cubes, regardless of colour, corresponded to the denominator. The students started to build bars corresponding to fractions like $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{5}$. Figure 1 and 2 show some examples of such bars. When the students were acquainted with how the bars could be used to depict different fractions, they started to order two fractions with different denominators by building bars that corresponded to the two fractions.

SOME DEFINITIONS

In this section we will define the concepts *strip*, *length*, *height* and *congruent bars*. The word “*strip”* was used frequently by the students. The words *“length”, “height”* and “*congruent bars”* were not used by the students, but we needed those terms in order to be able to analyse and communicate the students’ strategies. These definitions presuppose that the bars are oriented in the same way as in figure 1. A *strip* is a bar with *height* 1. The left part of figure 1 shows a strip that corresponds to $2/5$. We say that the *length* of this strip is 5 because it consists of 5 cubes. If a fraction were to be expanded, the students usually increased the *height* of the bar. The bar to the right in figure 1 is made up of two strips, and this bar corresponds to the fraction $2/5$ expanded by 2. The height equals the expansion factor which is 2. When we use the concept *physical* length or height, we do not mean the number of cubes, but the physical measure of the corresponding distance. Two bars are said to be congruent if the corresponding rectangles are congruent, regardless of the colour of the cubes.

Figure 1: To the left is a strip that corresponds to $2/5$. To the right is a bar that corresponds to $2/5$ expanded by 2.

THE TRIAL AND ERROR STRATEGY TYPE

The students were shown a rectangular piece of cardboard, and they were asked the following question: "If this was a real chocolate bar, and you could choose between $2/5$ or $1/3$ of the whole bar, what would you choose?" They were not able to answer. Then they were given a task that instructed them to build some bars where $2/5$ of the cubes were brown and put them in a heap, and to build some bars where $1/3$ of the cubes were brown and put them in another heap (see figure 2 for an example of such bars). Finally they were instructed to find two congruent bars, one from each heap, and count the brown cubes in the two congruent bars. In this way they found out that $2/5$ is greater than $1/3$. This strategy corresponds to an elementary level of objectification because it is a trial and error strategy, and the students focus on building two congruent bars.

Figure 2: Bars used to order $2/5$ and $1/3$.

THE FACTUAL STRATEGY TYPE

In this section we will describe the factual strategy of expanding two fractions to a common denominator, which was frequently used by the students. In the next section we will account for the common features between this type of strategy and Radford's factual level of generality. The factual strategy was invented by the students without any influence by the researcher, and it was more effective than the labour-intensive trial and error strategy. We will now give an example of this type of strategy based on an excerpt selected from one of the groups when they were working on the following task:

Make a chocolate bar where $\frac{3}{5}$ of the chocolate is brown, and make another where $\frac{2}{3}$ is brown. The bars are to have the same size. Which of the bars have most brown chocolate? Which of the fractions $\frac{3}{5}$ and $\frac{2}{3}$ are the biggest?

Mary has built two congruent bars, and she was asked to pretend that she was a teacher and explain to the others what she had done.

Mary: First you build a strip with three fifths *(showing a strip that corresponds to* $\frac{3}{5}$*, picture 1)*. Then you build on as much as you think it shall be. If I for example build four of these *(picture 2)*. Then you build two thirds *(showing a strip that corresponds to* $\frac{2}{3}$*, picture 3)* and see whether it fits or not *(picture 4)*. So now you have found out how it should fit *(removes one of the four strips so that the bar corresponding to* $\frac{3}{5}$ *consist of three strips, picture 5)*. Then you enlarge it *(expanding the strip that corresponds to* $\frac{2}{3}$ *so that it gets the same size as the bar corresponding to* $\frac{3}{5}$ *, picture 6).*

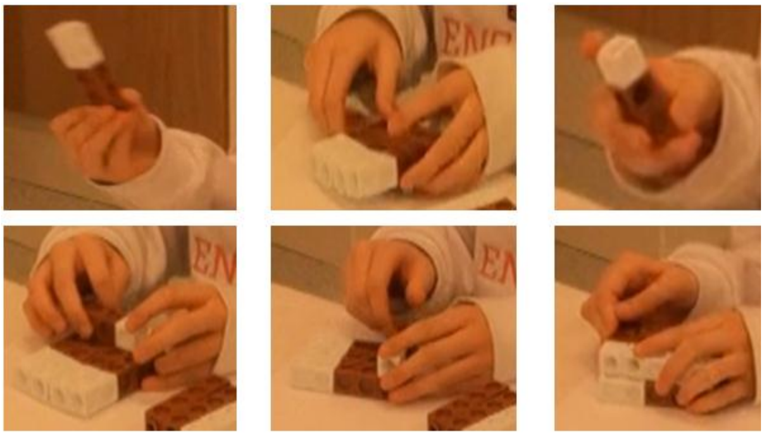

Figure 3: Picture 1 to 3 is on the first row, and picture 4 to 6 on the second.

Mary starts with a strip that corresponds to $\frac{3}{5}$. Then she expands this strip so that the height of the resulting bar becomes 4. She places the strip that corresponds to $\frac{2}{3}$ on the top of this bar, and she finds out that she has to remove one of the strips to obtain the right height. Finally she places the strip that corresponds to $\frac{2}{3}$ on the top of the bar that corresponds to $\frac{3}{5}$ and she expands this strip until the two bars become congruent. An important element of the procedure on this stage of the objectification process is to find the physical heights of the two congruent bars, and this is done through the physical lengths of the two strips that correspond to the fractions that are to be expanded.

According to Radford (2010a), there are different layers of objectification in the learning process, and in connection with these layers different aspects of the mathematical object is at the centre of the students' attention. In the factual strategy, the students focus on the physical lengths of the strips that correspond to the fractions that are to be expanded. This means that a deeper layer of the mathematical structure in the procedure of expanding two fractions has become apparent to them, and thus they seem to have reached a deeper layer of objectification.

In the factual strategy, the physical heights of the two congruent bars are not found through the denominators of the fractions that are to be expanded, but through the physical lengths of the two strips that correspond to these fractions. The physical heights of the bars are variable quantities in the expansion procedure which are not enunciated, but they are expressed through actions. No mathematical symbols were used in the process of expanding the two fractions. In the next section we will argue that these properties of the factual strategy have several common features with the factual level of generality introduced by Radford.

RADFORD'S FACTUAL LEVEL OF GENERALITY

We will now give a short description of the factual level of generality that is reported by Radford (2006). The students he referred to were to generalise a number pattern that was expressed by a visual representation, see figure 4.

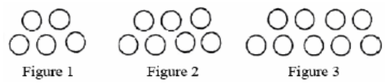

Figure 4: One of the patterns in Radford's studies.

An example of a way to determine the number of circles in a figure on the factual level of generality was (Ibid.): "One plus one plus three, two plus two plus three, three plus three plus three (p. 11)." Here the circles were grouped the following way by pointing gestures of the student, see figure 5:

Figure 5: Grouping of circles on the factual level.

No mathematical symbols were used in this generalisation. The crucial element of the generalisation of the number pattern is to express the variable quantity, i. e. the figure number, in some way. On the factual level of generality this quantity is not articulated in a direct way, but it is expressed in concrete actions (Ibid.): "[…] in factual generalizations, *indeterminacy* […] does not reach the level of enunciation: it is *expressed in concrete actions* [...] (p. 9)." The factual strategy reported in the previous section resembles Radford's factual level of generality. The physical heights of the two congruent bars are variable quantities in the expansion procedure which are not explicitly articulated, but expressed through actions. No mathematical symbols were used in the expansion process.

THE CONTEXTUAL STRATEGY TYPE

In this section we will delineate the contextual strategy of expanding two fractions to a common denominator which was often used by the students. In the next section we will elucidate the common features between this type of strategy and Radford's contextual level of generality. The contextual strategy was discovered by the students without any influence by the researcher, and it was an improvement of the factual strategy. The following task was given to the students:

> Make a chocolate bar where $\frac{2}{3}$ of the chocolate is brown, and make another where $\frac{5}{7}$ is brown. The bars are to have the same size. Which of the bars have more brown chocolate? Which of the fractions $\frac{2}{3}$ and $\frac{5}{7}$ are the biggest?

Cathie has built two congruent bars, and she has written down an explanation as to how she built the bars which she was asked to read aloud.

Cathie: First you make a strip with five sevenths *(picture 1)*. Then you make another one that shall be two thirds *(picture 2)*. Then you see that on two thirds, that the bottom number is three *(pointing gesture with the pencil on a strip corresponding to* $\frac{2}{3}$*, picture 3)*. So you build three strips with five sevenths *(pointing gesture with the pencil, picture 4)*. Then you see that the bottom number in five sevenths is seven. Then you take seven lengthwise *(gliding pointing gesture with the pencil, picture 5)*.

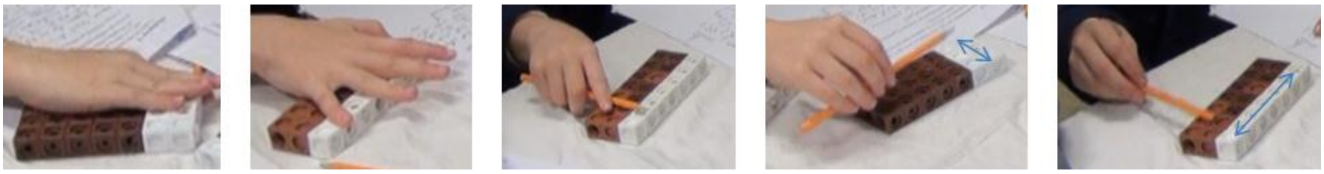

Figure 6: Picture 1 to 5 is ordered from the left to the right.

Cathie starts with a strip that corresponds to $\frac{5}{7}$, and she expands this strip so that the height of the resulting bar becomes 3 because the denominator of the other fraction is 3. Then the strip that corresponds to $\frac{2}{3}$ is expanded so that the height of the resulting bar becomes 7 because the denominator of the other fraction is 7. In the factual strategy the heights of the two congruent bars were found through the physical lengths of the two strips that corresponded to the fractions that were to be expanded. In the contextual strategy these heights were found through the denominators of the fractions which the students referred to as "the bottom numbers". The students focused on the denominators of the fractions that are to be expanded. This means that a deeper layer of the mathematical structure in the procedure of expanding two fractions to a common denominator has become apparent to them, and thus they seem to have reached a deeper layer of objectification.

RADFORD'S CONTEXTUAL LEVEL OF GENERALITY

We will now shortly describe the contextual level of generality that is reported by Radford (2006). An example of a contextual generalisation of the number pattern shown in figure 4 is (Ibid.): "You have to add one more circle than *the number of the figure* in the top row, and add two more circles on the bottom row (p. XLI)." The pivotal element of the generalisation of the number pattern is to express the variable quantity, i. e. the figure number, in some way. On the contextual level of generality, this quantity is reaching the level of enunciation. In the example above this was done through the formulation *"the number of the figure".* According to Radford (2010b), the explicit mentioning of the figure number is a hallmark of the contextual level of generality:

> The indeterminate object variable is now explicitly mentioned through the term "number of the figure." However, […] the new form of algebraic thinking is still contextual and "perspectival" in that it is based on a particular way of regarding something (p. XLI).

What characterises the contextual level of generality? Still mathematical symbols are not used in the generalisation. Like the factual strategy, the contextual strategy also originates from a visual approach to the process of generalisation, but now actions and gestures are less prominent. In addition to this, the variable quantity is explicitly mentioned. The contextual strategy reported here shares these characteristics. The denominators of the two fractions which are two variables in the expansion procedure equalling the expansion factors, are now explicitly articulated through the words "the bottom numbers". The expansion is not carried out by mathematical symbols, but through actions, namely a building procedure, although the actions are less prominent.

THE EMBODIED-SYMBOLIC STRATEGY TYPE

On this stage of the objectification process, the students have started to add fractions with different denominators, and the researcher has shown them how such calculations can be written down with mathematical symbols. In the excerpt we will now analyse, the following task was given to the students:

> One Saturday Peter makes a pizza for himself and his friends. When they have eaten, there is $\frac{1}{3}$ pizza left which he puts in the freezer. The next Saturday he also makes a pizza for his friends. Then there is $\frac{2}{5}$ pizza left which he puts in the freezer. How much pizza has Peter frozen after the two Saturdays?

Cathie has solved the task by building two congruent bars, and she has also carried out the calculation by writing down mathematical symbols, see figure 7. Then she was asked to explain what she had done.

$$\frac{1}{3}+\frac{2}{5}=\frac{1\cdot 5}{3\cdot 5}+\frac{2\cdot 3}{5\cdot 3}=\frac{5}{15}+\frac{6}{15}=\frac{11}{15}$$

Figure 7: Cathie's calculation of $\frac{1}{3}+\frac{2}{5}$.

Cathie: First I wrote one third plus two fifths. Then the equal sign. Then I wrote one third again with a long fraction line. Then I counted how many I had to expand it to, which was five *(counting on the bar that corresponds to $\frac{1}{3}$, picture 1).* Then I wrote times five over and under the fraction line. Then plus two fifths. Then I counted how many strips I had expanded it to *(gliding pointing gesture along the bar that corresponds to* $\frac{2}{5}$*, picture 2)*, which was three. Then I multiplied that *(pointing at* $\frac{1\cdot5}{3\cdot5}$ *on her sheet)* which became five fifteenths. Plus that *(pointing at* $\frac{2\cdot3}{5\cdot3}$ *on her sheet)* which was six fifteenths. Which equals eleven fifteenths.

Figure 8: Picture 1 and 2 is ordered from the left to the right.

In the factual and contextual strategy the students' attention were directed at how to build two congruent bars, but now the building procedure is no longer in focus, and the students gave no explanation as to how they built the congruent bars. Instead their attention was directed at the connection between the bars and the mathematical symbols. A new aspect of the expansion procedure had become apparent to them, and consequently they seem to have reached a deeper level of objectification. The expansion factors in the symbolical representation of the procedure are now found through the heights of the two congruent bars. The students first had to build the two congruent bars that corresponded to the fractions that were to be added, and then they were able to write down the symbolic representation of the procedure.

THE SYMBOLIC STRATEGY TYPE

After some time the students discovered that it was not necessary to build bars in order to add fractions with different denominators. The following task was given to the students:

> Each of the two brothers Bill and Benny won a chocolate bar at the charity bazaar. Bill gave away $\frac{2}{3}$ of his chocolate to his mother, and Benny gave away $\frac{1}{5}$ of his chocolate to his mother. How much chocolate did their mother get?

Peter had on his own initiative solved the task without using the multilink-cubes, and he had written down the mathematical symbols shown in figure 9. He had also written down an explanation of what he had done and was asked to read it aloud.

$$\frac{2}{3}+\frac{1}{5}=\frac{2\cdot5}{3\cdot5}+\frac{1\cdot3}{5\cdot3}=\frac{10}{15}+\frac{3}{15}=\frac{13}{15}$$

Figure 9: Peter's calculation of $\frac{2}{3}+\frac{1}{5}$.

Peter: First I wrote two thirds plus one fifth which equals two thirds with a long fraction line. Then I put times five on top and bottom because the denominator of the other fraction was five. Then I wrote plus one fifth with

> a long fraction line, times three because the denominator of the other fraction was three.

The students' attention was removed from the interplay between the bars and the mathematical symbols, and they focused only on the symbolic representation of the calculation. Consequently they seem to have reached a deeper layer of objectification.

RADFORD'S SYMBOLIC LEVEL OF GENERALITY

On the symbolic level of generality, a formula for the number of circles in a figure is obtained. In connection with the number pattern that is shown in figure 4, an example of a symbolical generalisation is $2n+3$. About this level of generality Radford says (2010a): "The understanding and proper use of algebraic symbolism entails the attainment of a disembodied cultural way of using signs and signifying through them (p. 56)." The symbolical strategy described in the previous section also "entails the attainment of a disembodied cultural way of using signs". The expansion is carried out by mathematical symbols, and there is no longer any reference to the building process.

CONCLUSION AND FURTHER RESEARCH

Our starting point was the following research questions: "Which strategies do the students employ as they expand two fractions to a common denominator, and what aspects of the expansion process is at the centre of the students' attention in these strategies?" At different stages of the objectification process, different aspects of the expansion procedure have been at the centre of the students' attention. In the factual strategy the students focused on the physical lengths of the strips that corresponded to the fractions that were to be expanded. These physical lengths were used to find the physical heights of the two congruent bars that were central in the process of expanding the two fractions. In the contextual strategy, the students' attention was directed at the denominators of the two fractions. These denominators were used to find the heights of the two congruent bars. In the embodied-symbolic strategy the students focused on the interplay between the two congruent bars and the mathematical symbols used to represent the procedure. The expansion factors in the symbolic representation of the procedure were found through the heights of the two corresponding congruent bars. In the symbolic strategy, the bars were no longer used, and the students' attention was directed at the mathematical symbols representing the calculation. In this case the expansion factors were found by the denominators of the fractions. These strategies correspond to different levels of objectification, and on these levels the students relate to the structure of the mathematical object – "the expansion of two fractions to a common denominator" – in more sophisticated ways.

Radford has introduced the factual, contextual and symbolic layer of generality concerning generalisation of number patterns. In this paper these concepts have been generalised to the students' strategies of expanding two fractions to a common denominator. In addition we have also described the embodied-symbolic strategy. Our hypothesis is that the type of strategies reported can be used as an analysing tool in connection with other artefacts and other mathematical subjects. The common features between three of these strategies and Radford's levels of generality confirm this hypothesis, but there is a need for more research to elucidate to what degree the four strategies we have described are instances of four general types of strategies that might be applied to other fields.

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