MENTAL COMPUTATION STRATEGIES IN SUBTRACTION PROBLEM SOLVING

Cristina Morais

Institute of Education of Lisbon University (PhD student)

Lurdes Serrazina

Escola Superior de Educação, Instituto Politécnico de Lisboa

This paper refers to part of a qualitative study made by the first author that had the main goal to understand which mental computation strategies are used by first grade pupils in addition and subtraction problem solving, namely to understand how the different addition and subtraction semantic contexts influence the mental computation strategy used in its resolution.

In this paper, we will present and discuss the strategies used at different subtraction problem situations by a pupil that constituted one of the three case studies held in the large study.

Keywords: number sense, mental computation, subtraction, strategies, problem solving.

INTRODUCTION

Mental computation is closely connected to one major goal in mathematics education of the elementary years: the development of number sense (e.g. NCTM, 2007).

Sowder (1992) associates number sense to an intuition and defines it as a well organized, conceptual network that allows the relationship between numbers, operations and its properties, and a flexible and creative way of problem solving. Similarly, Dehaene (1997) refers to number sense as an intuition about numerical relations, describing it as “a short-hand for our ability to quickly understand, approximate, and manipulate numerical quantities” (Dehaene, 2001, p.17).

McIntosh, Reys and Reys (1992) describe number sense as:

> “a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations” (p. 3).

Thus, in addition to other aspects, a good number sense implies a thorough and flexible understanding of numbers and their relationships, essential for the development of efficient and useful strategies, like mental computation strategies.

THEORETICAL FRAMEWORK

The importance of mental computation in the development of number sense is highlighted by several authors, but what is mental computation?

Buys (2008) briefly describes mental computation as "moving quickly and flexibly through the world of numbers" (p. 122) and characterizes it as a computation involving: a) numbers and not digits; b) computation properties and number relationships; c) a good understanding of numbers and a thorough knowledge of elementary facts up to twenty and up to one hundred; and d) the use of intermediate notes, but mainly calculating mentally. Verschaffel, Greer and De Corte (2007) add that "it's not the presence or absence of paper and pencil, but rather the nature of the mathematical entities and actions that is crucial in our differentiation between mental arithmetic and (written) algorithms" (p.566). Should be noted that this is the understanding of mental computation we will consider throughout the paper.

The subtraction strategies used by children depend on and evolve from strategies used in this operation with numbers smaller than twenty (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema, 1997). Thus, in this field of numbers, Thompson (2009) refers the following subtraction strategies: i) count out; ii) count back from; iii) count back to; iv) count up; and v) use known subtraction facts and use derived facts. Included in the strategies with known numeric facts, the author emphasizes the importance of strategies of jumping via ten.

For subtraction of numbers higher than twenty, in Dutch literature different types of strategies[1] are identified, organized into two categories: N10 and 1010 (e.g. Beishuizen, 2009).

In the N10 category (number+10 or number-10), at the first number is subtracted a multiple of 10. In this category, distinguishes itself a more complex strategy, N10C (compensation), in which at the first number is subtracted a multiple of 10 approximated to the second number to facilitate the computation. The result is then compensated. Another type of strategy, belonging to the N10 category, is identified as A10 (adding on). In this strategy at the first number is subtracted a part of the second number, so that the result is a multiple of 10. Then, the remaining part of the second number is subtracted.

In 1010 category, numbers are split into tens and units that are subtracted separately and the final result is obtained through rearrangement of the number. A 1010's variant is 10S (sequencial), in which the numbers are initially split into tens and units that are subtracted sequentially. Beishuizen (2009) refers that 1010 strategy may cause conflict in computations like 74-38, because pupils cannot be able to solve 4-8 and wrongly compute 8-4. The author adds that the difficulty of this type of strategy

[1] Conscious that perhaps we are using the term "strategy" to what Beishuizen refers as computation procedures, in this paper, this term refers to the strategies as N10, 1010 and its variants.

it's not the decomposition procedure but the correct rearrangement of numbers. According to the same author, N10 strategy is less vulnerable to these mistakes, so it is more efficient. However, its use requires a good ability when subtracting multiples of 10 from any number.

According to empirical research data presented by several authors (e.g. Beishuizen, 2001; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Thompson & Smith, 1999), pupils seem to prefer strategies from N10 category when solving subtraction computations. Furthermore, the success when using N10 strategies to solve subtraction computations is higher than with 1010 strategies. This last aspect, as Beishuizen (2001) states, seems to confirm the frailty of 1010 strategies, particularly regarding the lost of number sense while using the strategy.

In the elementary years, contexts provide the basis for computation (Treffers, 2008) and the support of the pupils' thinking (Ministério da Educação, 2007). For this reason, different subtraction contexts were chosen to provide the basis for the use and development of mental computation strategies.

There are different subtraction situations and, in this paper, we consider the situations presented by Ponte and Serrazina (2000): i) *take away*: part of a quantity is removed; ii) *compare*: two quantities are compared in order to find the difference between the two; and iii) *complete*: a value is found in order to add to a quantity so that a specific number is obtained.

METHODOLOGY

This paper refers to part of an empirical research made by the first author, as part of a master's dissertation, that had the main goal to understand how first grade pupils develop mental computation strategies, in an addition and subtraction problem solving context. To do that tried to answer three questions: a) Which mental computation strategies do pupils use when solving addition and subtraction problems?; b) How do these strategies evolve?; and c) Do the addition or subtraction problem situations influence the mental computation strategy used in its resolution?

In this paper, we will focus on mental computation strategies used in different subtraction problems.

The study was a qualitative one and three case studies were carried out. Data were collected by the first author in her first grade class, in a private school in Lisbon. Two problem chains[2] were solved in pairs[3] by pupils, between January and June 2010. All

---

[2] The word "chain" is use to identify the set of problems that were developed by the researcher and solved by the pupils. These sets of problems are identified as a problem chain because they were design to cover all the different subtraction problem situations. Also, the numbers involved in each problem were thoroughly selected so that they were progressively higher, increasing the difficulty of the computations.

problem solving lessons had the following moments: i) presentation of the problem, in which it was read by a pupil and possible doubts were clarified; ii) solving the problem in pairs; iii) presentation and discussion of the most significant solving strategies for the whole class; and iv) overview and identification of the most efficient strategies. A third and final problem chain was solved individually and outside the classroom by the three pupils who constituted the case studies, at the beginning of the second grade, in October 2010.

The study data were collected using video and audio recording (data from all lessons was fully transcribed), participant observation, pupils' records and field notes.

In the three problem chains there were 13 subtraction problems, covering the different subtraction problem situations. Table 1 shows the evolution of the magnitude of the numbers selected for each problem, as well as the presence of subtractions with and without regrouping, and the operations with values involving a different digit number. It is important to note that in the Portuguese curriculum there is no limit for the magnitude of numbers that should be worked in the first grade, and that, along the study, numbers were selected depending on the pupils' progress.

Problem chain	Subtraction situations	Computation	
1	*Compare* a)	20-6	subtraction
	Take away	15-7	subtraction
	Complete a)	28-16	subtraction without regrouping
	Complete	25-18	subtraction with regrouping
2	*Complete*	55-32	subtraction without regrouping
	Take away	49-26	subtraction without regrouping
	Compare	42-14	subtraction with regrouping
		75-48	subtraction with regrouping
	Take away a)	82-36	subtraction with regrouping
	Complete	124-47	subtraction with regrouping
3	*Compare* a)	157-43	subtraction without regrouping
	Take away	257-125	subtraction without regrouping
	Complete a)	250-135	subtraction with regrouping

a) Problems that will be described and discussed in this paper.

Table 1: Computations involved in the subtraction problems (presented by temporal order)

[3] The pupils' pairs varied according to how the work was usually developed in the classroom.

Content analysis was done and the categorization of computation strategies with numbers up to twenty referred by Thompson (2009) were followed, as well as the mental computation strategies with numbers higher than twenty identified by Beishuizen (2001, 2009) and Beishuizen and Anghileri (1998).

For this paper we selected five problems (indicated in Table 1) to describe and discuss the computation strategies used by Cátia, one of the three studied pupils,. The choice was made to include the different semantic situations and also regrouping/not regrouping and magnitude of the numbers.

The transcriptions that will be presented were from the audio and video recordings of the lessons. During the lessons, the teacher sometimes approached Cátia and her colleague, questioning the strategy used, like she did to the other pairs of pupils in the class.

RESULTS

Problem one, 1st chain *(compare)* – The sister of Leonor and Rita is 20 years old. How many years older is she? (Leonor and Rita are twins from the class and are 6 years old.)

This was the first problem with a *compare situation* solved in the study. Cátia uses a jumping via ten strategy, adding 6+4=10, and then she uses basic number facts to reach 20. Finally, Cátia adds the partial results (figure 1), as she explained:

Cátia:	I did like this… I know that 6 plus 4 is 10. Then I did a jump of 3 that was 13.
Teacher:	Why did you make a jump of 3? Why not a jump of 4 or 2…?
Cátia:	Because… I decided to do one of 3 because I thought it was a good computation to do. Then I did 4 plus 3 that is 7. Then I did another jump of 3, that it was as if this 1 [from 13] didn't exist. Then I did 7 plus 3 that was 10, and then it was just plus 4 and 10 plus 4 is 14.

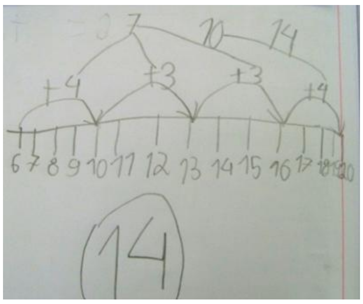

Figure 1. Cátia's solution of problem one

Problem two, 1st chain *(complete)* - Marta is reading a book. She has already read 16 pages and the book has 28. How many pages are left?

Cátia uses an additive A10 strategy, approaching 16 to a multiple of 10, 20. Then, she adds 8 (20+8), which is a basic number fact for her.

Cátia explained her strategy to her colleague:

> Cátia: Pretend that 16 was 6, and 20 was 10. I know that 6 plus 4 is 10. Then I did a jump to 28 and saw it was a jump of 8. And 8 plus 4 is 12.

Cátia didn't feel the need to write down all the numbers in the number line, she just marked the numbers of her computations (figure 2).

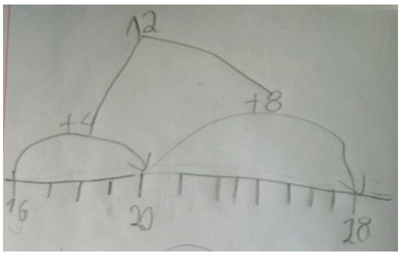

Figure 2. Cátia's solution of problem two

After solving the problem, Cátia decided to verify the result. She used an additive 1010 strategy, which she had used for the first time in a previous addition problem (figure 3).

Cátia shows understanding and ease using this type of strategy, also demonstrating a good comprehension of the problem and of the relation between addition and subtraction.

> Cátia: It had to be 12 plus 16, because we thought it was 12. We wrote 12 and 16. Then we took the 10 of the 12 and in the 10 of the 16, which were 20. Then 6 plus 2 was 8.

Figure 3. Cátia's verification strategy of the result of problem two

Problem three, 2nd chain *(take away)* - Leonor and Simão are playing a boardgame. Leonor is in house number 82. In that house she reads "How unlucky! You have to go back 36 houses." In which house is she now?

Cátia and her colleague identified easily the subtraction involved in the problem. Once again, Cátia uses a subtractive1010 strategy and subtracted without difficulty 80 minus 50. In the subtraction 2 minus 6, she splits the 6 into 2 plus 4:

Cátia: I've splitted the 6 into 2 plus 4, and 2 minus 2 equals zero… I've already spent this 2. Then, zero minus 4 is negative 4.

She seems to overcome easily the difficulty in 2-6. To confirm the result, Cátia uses a subtraction A10 strategy, supported by the empty number line (figure 4).

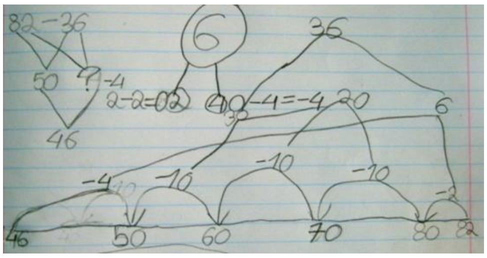

Figure 4. Cátia's solution of problem three

Problem four, 3rd chain *(compare)* - Miguel and Cláudia are playing the "Stop or Go" game. At the end, Cláudia got 157 points and Miguel got less 43 points than her. How many points did Miguel get?

Cátia starts to solve the problem using a subtractive 1010 strategy. She started to subtract the tens, then the units and finally the hundreds. However, Cátia did 100 minus 14. Unsure of the result, she added 86+43 using an additive 1010 strategy, obtaining 129.

Then, she recalculated 157 minus 43, now adding the partial differences, 100 and 14 (second computation from the right, in figure 5), explaining:

Cátia: I tried…but I didn't know if I had to add or to subtract [first computation from left, in figure 5]. I've subtracted but it wasn't right. Now I'm doing it again and now it's right.

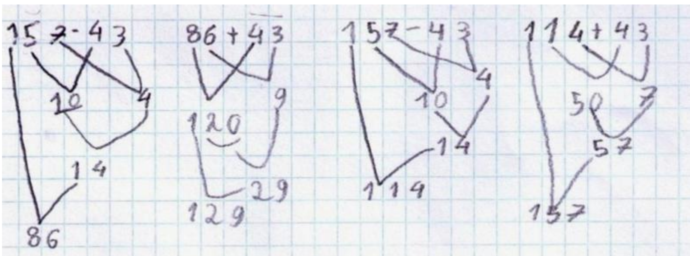

Figure 5. Cátia's attempts when solving problem four

Although Cátia reveals difficulty when rearranging the number after using a subtractive 1010 strategy, she also shows a good comprehension of the relation between addition and subtraction, using that knowledge in order to overcome her difficulty.

Problem five, 3rd chain *(complete)* - Leonor went to a bookstore. That bookstore was doing a competition: the customer number 250 who enters the store wins a book collection of his choice! Leonor was the customer number 135. How many customers must enter the store until the prize is awarded?

Cátia uses an additive A10 strategy, adding parcels to 135, in order to approach 250 (figure 6). In these partial additions, Cátia approximates the partial results to reference numbers, using basic fact numbers.

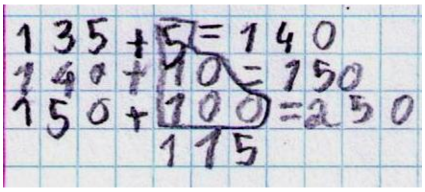

Figure 6. Cátia's solution of problem five

DISCUSSION

As seen in the results described above, the *compare* and *complete* problems were generally translated by Cátia as an expression like a+?=b, and solved mainly through additive A10 strategies. These results are consistent with the findings of several studies (Carpenter et al.,1998; De Corte & Verschaffel, 1987; Heirdsfield & Cooper, 1996) that identify this type of strategy as the most frequently used by children when solving this kind of subtraction problems.

Cátia uses a subtractive 1010 strategy in the *take away* problem of the second chain and in the *compare* problem of the third chain. In the fourth problem (*compare* problem) it is possible to identify the weakness of this type of strategy in subtractions with values represented with a different digit number, which led to an incorrect rearrangement of the final result. As Beishuizen (2001) stresses, this fact is due to the lost of number sense during the computation. However, Cátia overcame this difficulty through her critical analysis towards the result, using the relationship between addition and subtraction to verify the result.

In the *take away* problem, Cátia used the subtractive 1010 strategy with comprehension and without difficulty even in those situations that could cause conflict (e.g. 82-36). In the other *take away* problems, from the second and third chain, in the large study, Cátia had used this strategy with the same easiness, what seems to indicate the comprehension that she has about subtraction, particularly about the lack of commutativity of this operation. It also seems to show her understanding and mastery of negative numbers, that Thompson (2000) relates to the students with more proficiency at computation.

Note that the mental computation strategies used by Cátia when solving the subtraction problems are associated in the literature to older pupils (e.g. Beishuizen, 2001; Buys, 2001; Cooper, Heirdsfield & Irons, 1995; Thompson & Smith, 1999). Maybe this was due to a good learning environment that these children had, which allowed them to develop a good number sense.

REFERENCES

Beishuizen, M. (2001). Different approaches to mastering mental calculation strategies. In J. Anghileri (Ed.), *Principles and Practices in Arithmetic Teaching – Innovative Approaches for the Primary Classroom*, pp. 119–130. Buckingham: Open University Press.

Beishuizen, M. (2009). The empty number line as a new model. In I. Thompson (Ed.), *Issues in Teaching Numeracy in Primary schools*, pp. 157–168. Open University Press.(Reprint of 1999)

Beishuizen, M. & Anghileri, J. (1998). Which mental strategies in the early number curriculum? A comparison of British ideas and Dutch views. *British Educational Research Journal*, *24*(5), 519-538.

Buys, K. (2008). Mental Arithmetic. In M. van den Heuvel-Panhuizen (Ed.), *Children Learn Mathematics: A Learning-Teaching Trajectory with Intermediate Attainment Targets for Calculation with Whole Numbers in Primary School* (pp. 121-146) Netherlands: Sense Publishers. (Original work published in 2001)

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema E. & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3-20.

Cooper, T. J., Heirdsfield, A. & Irons, C. J. (1995). Years 2 and 3 children's strategies for mental addition and subtraction. In B. Atweh & S. Flavel (Eds.) *Proceedings of 18th Annual Conference of Mathematics Education Research Group of Australasia* (pp. 195-202). Darwin: Mathematics Education Group of Australasia.

De Corte, E. & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education, 18*(5), 363-381.

Dehaene, S. (1997, October). What are numbers, really? A cerebral basis for number sense. *Edge, 28.* Recovered at November 26, 2012, of http://www.edge.org/documents/archive/edge28.html

Dehaene, S. (2001). Précis of the number sense. *Mind & Language*, *16*(1), 16-36.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. C. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 243-275). New York: Macmillan Publishing Company.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P. & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Heirdsfield, A. M. & Cooper, T. J. (1996). The "Ups" and "Downs" of subtraction: Young children's additive and subtractive mental strategies for solutions of subtraction word problems and algorithmic exercises. In P. Clarkson (Ed.), *Proceedings of the Nineteenth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 261-268). Melbourne: Deakin University Press.

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*(3), 2-8.

Ministério da Educação (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação: DGIDC.

Ponte, J. P. & Serrazina, M. L. (2000). *Didáctica da Matemática do 1º Ciclo*. Lisboa: Universidade Aberta.

Thompson, I. (2000). Mental calculation strategies for addition and subtraction: Part 2. *Mathematics in School, 29*(1), 24-26.

Thompson, I. (2009). Getting your head around mental calculation. In I. Thompson (Ed.), *Issues in Teaching Numeracy in Primary schools*, pp. 145–156. Open University Press.(Reprint of 1999)

Thompson, I. & Smith, F. (1999). *Mental calculation strategies for addition and subtraction of 2-digit numbers* (Report for the Nuffield Foundation), Department of Education, University of Newcastle upon Tyne.

Treffers, A. (2008). Grade 1 (and 2) – Calculation up to 20. In M. van den Heuvel-Panhuizen (Ed.), *Children Learn Mathematics: A Learning-Teaching Trajectory with Intermediate Attainment Targets for Calculation with Whole Numbers in Primary School* (pp. 43-60) Netherlands: Sense Publishers.(Obra original publicada em 2001)

Verschaffel, L., Greer, B. & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Reston, VA: NCTM.