SYNERGY BETWEEN VISUAL AND ANALYTICAL LANGUAGES IN MATHEMATICAL THINKING

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Visualization is a research field of growing importance in mathematics education. However, the study of its nature and relationship with other forms of recording and reporting information continues to be subject of reflection. In this paper we propose a way of understanding the language and the visual thinking, and their relationship with the language and analytical thinking, using the theoretical tools of the "onto-semiotic approach" of mathematical knowledge. By analyzing the mathematical activity deployed in solving a task, we show cooperative relations between the visual and analytic languages.

INTRODUCTION

Visualization has received much attention as a research topic in mathematics education, especially in the area of geometry (Bishop, 1989, Clement and Battista, 1992, Hershkowitz, and Van Dormolen Parzysz, 1996, Gutiérrez, 1996). Presmeg's work (2006) provides a comprehensive perspective of research on the role of visualization in teaching and learning mathematics in the International Group of PME. This survey concludes by stating 13 big research questions on this research field, and on which we focus on the following, in this paper: "What is the structure and what are the components of an overarching theory of visualization for mathematics education" (Presmeg, 2006, p. 227).

In this paper we are interested in advancing an answer to the problem of devising a theory to clarify the nature and components of visualization and its relationship to other processes involved in mathematical activity, their teaching and learning. A key aspect of developing a theory of visualization in mathematics education should include studying the relations of this form of perception with other ostensive modes of expression (in particular, sequential analytical languages), and especially its relation to non-ostensive mathematical objects (usually considered as mental, formal, or ideal objects).

As background on the problem of theoretical clarification of visualization in mathematics education we found Presmeg (2008), who expands the initial taxonomy suggested by Marcou and Gagatsis (2003) in terms of Peirce's triadic semiotics. Rivera (2011) analyzes the visual root of mathematical symbols and mathematical reasoning, and the implications of visualization to mathematics instruction.

THEORETICAL FRAMEWORK

As it can be read in Phillips, Norris & Macnab (2010) there is no clear consensus on how to define visualization, both in its role of process as object, and in its internal (mental) and external (perceptive) version. In our case we assume Arcavi's (2003, p. 217) proposal, which describes visualization in very general terms: "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". However, we consider necessary to deepen the distinction between visual and non-visual objects and processes, in order to study the necessary coordination between both types of objects in the construction and communication of mathematics.

We will analyze the notion of visualization by applying some tools of the "onto-semiotic approach" of mathematical knowledge (OSA) (Godino, Batanero, and Font, 2007). In this framework it is considered that the analysis of mathematical activity, the objects and processes involved in it, first should focus the attention on the practices carried out by people involved in solving certain mathematical problem situations. Applying this approach to visualization leads us to distinguish between "visual practices" and "non-visual practices" (symbolic/analytical practices), and to study the relationships between them. To this end we fix our attention on the kinds of languages and artifacts involved in a practice, which will be considered as visual if they put into play iconic, indexical or diagrammatic signs (Peirce, CP, 2.299).

Although symbolic representations (natural language or formal languages) consist of *visible inscriptions,* those inscriptions will not be considered as strictly visual, but as analytical or sentential. Sequential languages (e.g., symbolic logic, natural language) use only the relation of concatenation to represent relationships between objects. On the contrary, the diagrams use spatial relationships to represent the objects and relationships. "The idea is that sentential languages are based on acoustic signals which are sequential in nature, and so must have a compensating complex syntax in order to express certain relationships - whereas diagrams, being two-dimensional, are able to display some relationships without the intervention of a complex syntax" (Shin and Lemon, 2008, p.10).

Visual objects and visualization processes from which they come, form configurations or semiotic systems constituted by "the intervening and emerging objects in a system of practices, along with the interpretation processes that are established between the same (that is to say, including the network of semiotic functions that relate the constituent objects of the configuration)" (Godino et al, 2011, 255).

In the OSA it is highlighted the essential role of the ostensive dimension in mathematical practice when postulating that every mathematical object (abstract ideal, generally immaterial, not ostensive) has an ostensive facet, that is, publicly, visually, perceptually or otherwise demonstrable. This ostensive facet may consist of symbolic inscriptions, needed to represent the objects, understood as a unitary whole, and to be able to "operate" with them in progressive levels of generality, or using iconic or diagrammatic means that show the structure of the object, understood in a systemic way.

In the next section we show, by analyzing the solving of a task usually considered “of visualization" and using some onto-semiotic tools, that in mathematical activity participate, next to visual ostensive objects, other ostensive objects non visual (textual or analytic) to refer the non-ostensive objects involved (concepts, propositions, procedures, arguments). Both types of ostensive objects (visual and non-visual) play a role in the performance of mathematical activity, so that mathematics teaching should pay attention to the relationship between the two forms of semiotic representation of mathematical objects. Mathematical activity is analyzed theoretically and by using an example as being based on a variety of representational and linguistic or more general semiotic means. The goal is to highlight the intricate interplay of those means.

A MATHEMATICAL TASK AS A CONTEXT OF REFLECTION

In this section we discuss a mathematical task that uses visualization processes. The analysis of the proposed solution shows the network of visual and non-visual objects used, and the relationships established between them, namely, the semiotic system that forms. Briefly, the goal is to reveal the knowledge involved in the resolution and the synergy that exists between visual and analytical objects.

Statement: Write which of these figures represent the unfolding of a cube.

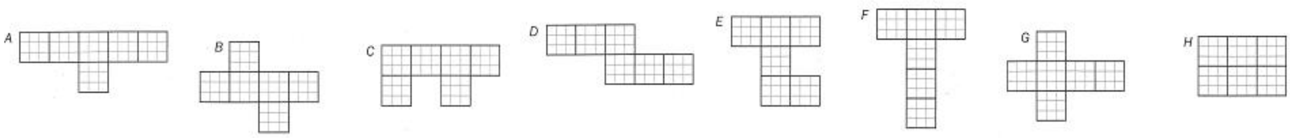


Figure 1: Potential unfolding of a cube

Solution:

The hexamine B, D, F and G correspond to a cube; in fact if we match (paste) the sides marked in figure 2 with the same symbols we get a cube.

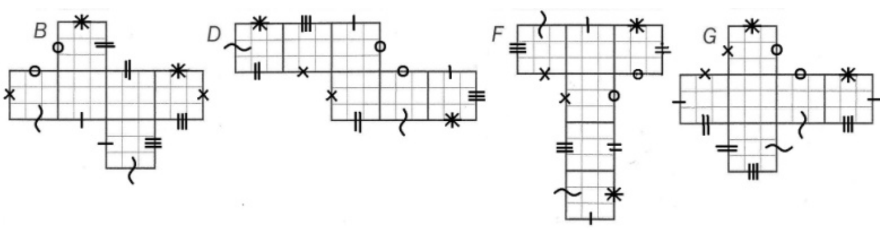


Figure 2: Unfolding of a cube

The remaining hexamine do not represent the unfolding of a cube. Indeed, the hexamine A, C and E do not correspond to a cube because of the overlap of two cube faces to fold the unfolding: in figure 3 we marked with the letters *a* and *b* the faces that overlap.

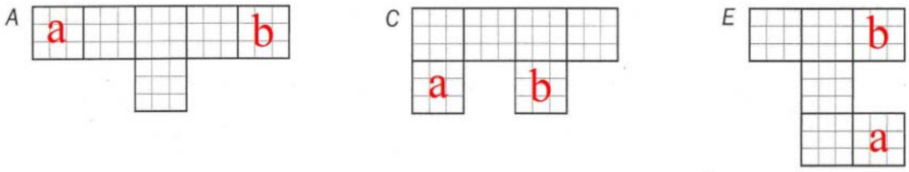


Figure 3: Faces that overlap

It can be visually tested (either empirically or mentally) the impossibility of closing the unfolding.

Onto-semiotic analysis of the solution

This is a visual mathematics task according to the characterization previously presented, but in justifying the answer, different analytical elements that are needed to prove its validity, emerge. To "see" the solution in figures 2 and 3 it is necessary to perform various visual operations and semiotic interpretations that may not be immediately perceived by the students. Such visual operations are supported by analytical elements that define the objects involved. For example, to justify that the sides marked with the same symbol overlap after the folding operation it must be made explicit the knowledge that the adjacent faces of a cube, by definition of this mathematical object, form a dihedral angle of 90°. Therefore, to construct the cube, the contiguous faces of the unfolding must rotate in the space 90º around the segments (which coincide with imaginary exes *x* and *y*). The direction of rotations varies as the considered unfolding.

The wording of the task is made up of linguistic elements (words) and visual elements that interact between them. The term "figure" refers to the drawings in figure 1: visual ostensive objects that have to be put in relation to the non-ostensive object "cube" regulated by a definition. To solve the task, the student must know the intuitive meaning of unfolding a polyhedron and recognize that the surface of the cube is developable. Intuitively, a surface is developable if it can be made from a Euclidean plane by "folding", which is "visually" manifested when it is possible to make appropriate models from a sheet of paper or flat cardboard. The development of the surface would be the plane figure obtained by the whole unfolding in the plane. This intuitive definition has to be applied to the particular case of the surface of a cube. The process of "unfolding" the surface of the cube in the plane can be represented visually (figure 4) and/or interpreted analytically.

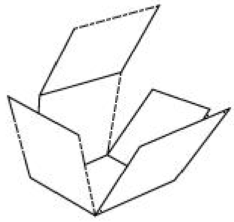


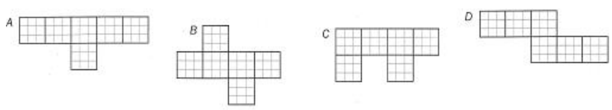
Figure 4: Possible way of unfolding a cube

Since the cube is a polyhedron with six square congruent faces, its unfolding is a set of six squares connected by one and only one side, so that: a) Each face of the cube corresponds to a single square of the development; b) It is possible to match all the sides of the squares that belong to the edge of the unfolding so that each pair corresponds to one and only one edge of the cube. Based on this last (analytical) condition it is argued (a visual way, figure 2) that the B, D, F and G hexamine represent unfolding of a cube, marking with the same symbol equivalent sides, that is, the pairs of external sides of the squares of the unfolding that are joined to form an edge in the cube.

In figure 3 it is visually argued that the hexamine A, C and E do not correspond to a cube due to superposition of a face when folding the unfolding. In fact the superposition of a face contradicts the (analytical) condition that each face of the cube corresponds to a single square of the unfolding. As regards the H hexamine it is showed that it does not represent an unfolding of a cube. This quite intuitive (and visual) statement can be justified asserting the impossibility of constructing a trihedral angle from four coplanar faces (fairly easy to visualize), or seeing that the representation contradicts the cube unfolding definition, since there are squares connected by more than one side.

This example shows the synergistic relationships between visual and non-visual objects in mathematical activity carried out to solve a problem of visual type. In particular, we can observe that the visual explanation of the task solution is supported by analytical elements related to the conceptual properties of the cube development. Some key semiotic functions involved in the visual justification of the task are summarized in Table 1.

Table1: Semiotic functions implicated in visual justification

Visual expression	**Analytical content**
"Cube", as mental image (non-ostensive visual object) that the person internally represents.	Concept of "cube" (Definition 1): polyhedron of six square congruent faces.
Figure 1: 	Concept of "unfolding of a cube" (Definition 2): Set of six squares connected by one and only one side, such that:

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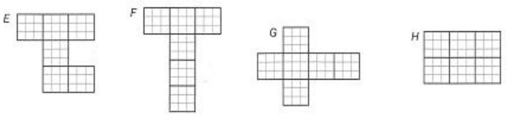
<td>

They are visual ostensive objects referring to potential unfolding of a cube.

The recognition of the unfolding of a cube can be accomplished through the visual operation "fold / unfold". The procedure may consist of the mental simulation of the physical action (non-ostensive visual process), be physically carried out (cut and paste) (ostensive visual procedure), or illustrated through visual language, as shown in Figure 4.</td>
<td>a) Each face of the cube corresponds to a unique square of the unfolding (property 1).
b) It is possible to match all the square sides belonging to the unfolding edge, so that each pair corresponds to one and only one edge of the cube (property 2).</td>
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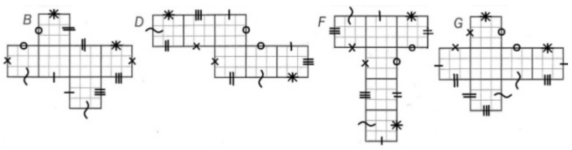
<td>If we fold the hexamine B, D, F and G along the sides and join together (physically or mentally) the sides marked with the same symbols, we obtain a cube (figure 2):


There are not overlap of faces and the unfolding closes. (Visual checking, ostensive or mental)</td>
<td>The hexamine B, D, F and G represent the unfolding of the cube, since it fulfills definition 2.

In particular, Figure 2 illustrates the analytical property 2.</td>
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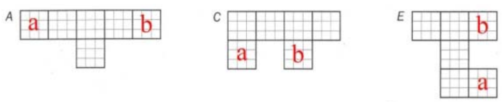
<td>In Figure 3 the faces marked with letters a and b overlap each other to perform the folding and one face would remain uncovered:


(Visual checking, ostensive or mental)</td>
<td>The hexamine A, C and E do not represent the unfolding of the cube because they do not respect the property 1.</td>
</tr>
<tr>
<td>It is visually verified (either ostensive or mentally) the failure of closing the unfolding.</td>
<td>The hexamine H is not an unfolding of a cube because it has squares connected by more than one side (it does not respects definition 2).</td>
</tr>
</table>

It is not always necessary to deploy the explicit analytical long speech that explains all the rules (concepts and propositions) which effectively support the justifications of the solutions. In this case, visual representations are revealed as a resource of effective expression to convince the reader that, indeed the hexamine B, D, F, and G correspond to the cube development, while this is not the case with hexamine A, C, E and H. But in any case, the rules defining the concepts and properties are still latent.

Fischbein (1993) notes that the mental transformations of three-dimensional objects are not only visual in nature (figural in the author's terminology): it is because we work with faces of a cube which has edges of equal size, faces which are square, angles which are right, and so on. "This is *tacit* knowledge that is involved in mental operations. Without this tacit conceptual control, the whole operation would have no meaning "(p. 159).

The delicate network of visual and non-visual ostensive objects, to refer to non-ostensive objects, always present in mathematical activity (cube concept, face, edge, vertex, and the related properties), and also for the effective realization of procedures and justifications, is put into effect not only with geometric tasks, but also with other mathematical contents. Godino, Gonzato, Cajaraville and Fernández (2012) analyze an algebraic task (proving that the sum of the first n odd numbers is n^2) with the support of visual representation, thus showing the same cooperative relations between the visual and analytical languages.

FINAL REMARKS AND IMPLICATIONS FOR MATHEMATICS EDUCATION

As conclusions of the analysis performed in this paper on visualization, we can say that the configuration of objects and processes used when carrying out a mathematical practice are the following:

(1) It always involves analytical languages in greater or lesser extent, although the task refers to situations on the perceptible world. This is essentially due to the regulatory-sentential nature of concepts, propositions and mathematical procedures.

(2) A non-visual task can be addressed, at least partially, through visual languages which enable to effectively express the organization or structure of the configuration of objects and processes used, especially with diagrams or with metaphorical use of icons and indexes.

Consequently, the configuration of objects and processes associated with mathematical practice will usually consist of two components, one visual and another analytic, which synergistically cooperate on the solution of the corresponding task (figure 5). The visual component can play a key role in understanding the nature of the task and at the time of making conjectures, while the analytical component will be in the moment of generalization and justification of solutions. The degree of visualization used in solving a task depends on the visual or non-visual character of the task and also on the subject's particular cognitive styles that resolved the task, as has been emphasized by several studies (Krutestkii, 1976; Presmeg, 1986; Pitta-Pantazi and Christou, 2009).

The analysis of visualization we have carried out, using some of the OSA tools, provides a complementary view regarding to other perspectives more focused on the description of visual/analytical cognitive styles and its influence on problem solving. Our goal has been to deepen into the nature of visualization and its relation to analytic-sequential forms of mathematical thinking. We sought to characterize mathematical practice in tasks involving visualization, whether performed by an individual (subjective knowledge), or shared in an institutional framework (objective knowledge), identifying the types of objects and processes involved in the performance of the practice.

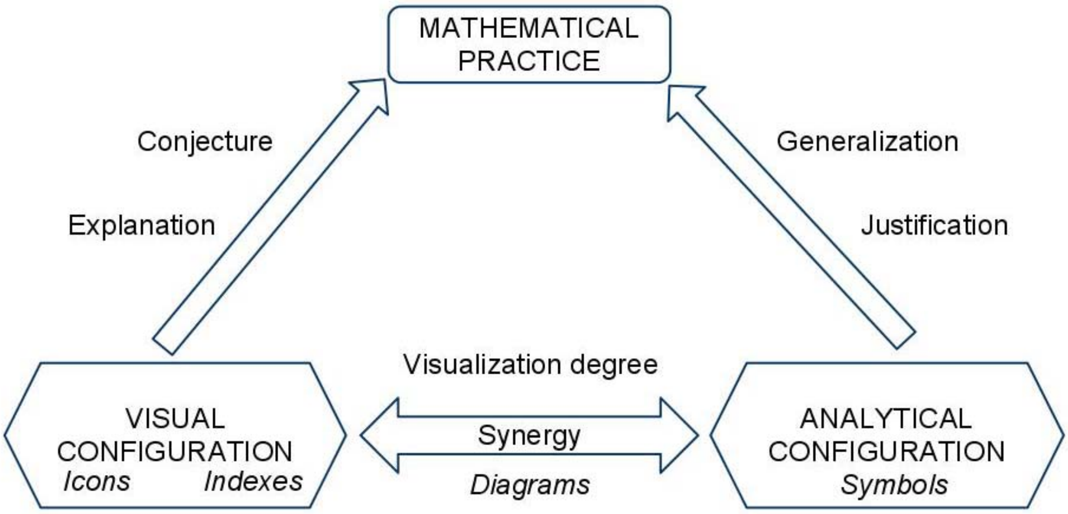


Figure 5: Synergy between visual and analytical configurations

A visual task can be tackled with analytical tools and vice versa, a non-visual task can be approached analytically with visual tools. Moreover, in conducting a visual practice non-visual objects are actually involved, and in the implementation of an analytical practice, visual objects, particularly diagrams, may be involved. This is a result of the implementation of the ostensive non-ostensive duality (Godino et al, 2011) to different types of mathematical objects, which carries out the dialectic between the visual and analytical. For any mathematical object the presence or intervention in its emergence and operation of an ostensive aspect (public, visible, symbolic or visual) and other non-ostensive aspect (rule, logic, ideal, mental) which interact in a synergistic way is postulated, as was shown in the analysis of the example in section 3.

An educational implication of our analysis is that subjects whose cognitive style is basically analytic (respectively, visual) should be instructed to develop visual skills (respectively, analytical), because both skills are useful for mathematical practice at different stages of their execution. Hence, it would be necessary to favor the development of the harmonic cognitive style described by Krutestkii (1976), which combines visual and analytical features.

It seems clear that visualization penetrates in all branches of mathematics, not only in geometry, in coordination with other forms of expression, especially analytical/ sequential languages. It is also present in the various levels of mathematical study, as well as in elementary as in higher education, or even professional. However, the analysis of the relative effectiveness of visual modes of reasoning regarding analytical modes, depending on the types of tasks and phases of study, is a subject that requires investigation. The interest of using iconic and diagrammatic representations has been generated by the assumption that somehow they are considered more effective than traditional logical representations for certain tasks. However, although there are some psychological advantages in using diagrams, they are often ineffective as representations of objects and abstract relations (Lemon and Shin, 2008).

The role of visualization in school or professional mathematical work is complex because it is often interwoven with the use of symbolic inscriptions, which although "are visible", their meaning is purely conventional. The problem is relevant even when the visualization referred to the use of visual objects, which interact not only with symbolic inscriptions, but also and mainly with the network of conceptual, procedural, and propositional objects that necessarily intervene in mathematical practice.

Teachers, curriculum developers and teacher educators should be aware of the role of visualization in building and communicating mathematics. On the other hand, one should not confuse the mathematical object with its ostensive representations, whether visual or otherwise. It is necessary to take into account the non-ostensive immaterial nature of mathematical objects and the complex dialectical relationships that are established between these objects and their material representations.

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