CLOSE YOUR EYES AND SEE…
AN APPROACH TO SPATIAL GEOMETRY

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In this paper we present an experiment that we conducted last spring (May 2012) in a high school in Sicily. The activities concerned some properties of quadrilateral and tetrahedron and aimed to introduce the study of spatial geometry by means of a correspondence that can be established between the two figures, by using suitable definitions. The experiment is part of a research centred on the investigation of visual challenges involved in doing spatial geometry and the role of technology to address these challenges.

Keywords: visualization, spatial geometry, dynamic geometry software

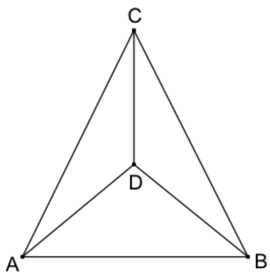
INTRODUCTION

1. "Good night mum. Please, leave the light on!", used to say little James every single night before falling asleep.

"Good night mum" said little James few nights ago… He was scared of dark. But all of a sudden, he realized that in the dark –well not dark dark!– *he could see*. In the dark, everything was clear. He did not need the night light on anymore.

2. It was June 2012 (La «scoperta» durante una gita del Liceo Galilei alla riserva del Plemmirio, *La Sicilia*, 11 Giugno 2012, p. 47) when Giorgia Florenza, a seventeen-year-old student, taking pictures during an excursion with her classmates near Siracuse, noted that there was *a face* in the rock she was taking a picture of: "With a simple click can come out prodigious things that sometime may escape to naked eye (authors' translation from the article).

3.



Teacher: And *what do you see* in the figure?

Student: A quadrilateral and/or a tetrahedron.

This is what a student said last May during a classroom activity in a scientifically oriented high school, in Giarre (Catania).

What one –being little James, Giorgia, or the student– sees or is able to see is *not* established *a priori* or universal, but it *may* depend on many different factors in many different contexts and experiences. It is situated and relative.

This paper presents a teaching experiment on Euclidean Geometry that we carried out last spring (May 2012) in a high school in Sicily. The experiment dealt with some properties of quadrilateral and tetrahedron, and aimed to introduce the study of 3D geometry through a correspondence established between those two figures, by using suitable definitions. This study is part of a wider research whose interests are on aspects related to visualization and visibility, which are crucial for students when dealing with the shape of space and three-dimensional objects.

THEORETICAL CONSIDERATIONS

> The fifteen-year-old students most repulsive subjects in mathematics were spatial geometry and statistics. Only ten percent of teachers taught spatial geometry. They said that they did not have enough time to teach it, but the real reason is that the students 'cannot see in 3D'. (Bakó, 2003, p. 1)

This quotation regards a study that was conducted in France, but it marks a delicate situation that is still recognized as widespread at school, no matter what country is taken. In fact, the teaching of spatial geometry is often neglected and pushed in the background and not given great priority (Mariotti, 2005; Villani, 2006; Oldknown & Tetlow, 2008), partly due to the poor knowledge and confidence of teachers about the topic, and partly to the general reputation that spatial geometry is difficult because it is difficult *to "see"* (among teachers as well as among students).

Nevertheless, 3D geometry plays an essential role in all scientific disciplines, from physics to astronomy and chemistry, from engineering to figurative arts, as well as in everyday life. So, it seems very important to regard the study of spatial geometry at school and the development of spatial sense as crucial for shaping the mathematical competence required to future citizens aware and able to make decisions. On the other hand, the National syllabus endeavours to promote it as an integral part of the mathematics teaching and learning since early grades. We agree with Villani (2006) that "the cultural and operational-technical worthiness of spatial geometry must prevail over the difficulties of its teaching in all school grades." (p. 66, authors' translation).

The question of *seeing* is certainly primary. 3D geometry involves visual challenges, mainly depending on the fact that we have to do with representations of geometrical entities, these being bi-dimensional representations of three-dimensional objects. In so doing, perception of the third dimension is not easy, and strongly implies issues of visibility and visualization (Hershkowitz, Parzysz & Van Dormolen, 1996; Ferrara & Mammana, submitted). Based on the double aspect of geometrical figures, that is, the distinction between their figural and conceptual properties (Fischbein & Nachlieli, 1998), the transition from the perceptual to the conceptual (from the drawing to the geometric figure) is even more difficult. As Rojano (2002) pointed out, in fact, a

> problem source for geometry students in their transition to the conceptual is the lack of previous visual education that can aid the systematization of their visual experiences, for instance, in the search for patterns or in the distinction between the role of drawings as geometric objects or as diagrammatic models of these objects. (Rojano, 2002, p. 153)

In a research of the late eighties, Parzysz (1988) already highlighted the presence of a "knowing vs seeing" conflict in the teaching of space geometry, which entails that

> The problems of coding a 3D geometrical figure into a single drawing have their origin in the impossibility of giving a close representation of it, and in the subsequent obligation of 'falling back' on a distant representation […] an insoluble dilemma, due to the fact that what one *knows* of a 3D object comes into conflict with what one *sees* of it. (pp. 83-84)

The relationship between vision and perception of space had also been discussed by Jules-Henri Poincaré (1905), who, in his famous book *Science and Hypothesis*, was taking care of the problem of conceptualizing space. He was arguing that space is not a pre-existing concept, our knowledge of it being instead determined by our way of being and staying in the world, that is, by our ordinary experience with space and three-dimensional objects. Our eyes form bi-dimensional images of the 3D world but

> sight enables us to appreciate distance, and therefore to perceive a third dimension. But everyone knows that this perception of the third dimension reduces to a sense of the effort of accommodation which must be made, and to a sense of the convergence of the two eyes, that must take place in order to perceive an object distinctly. (p. 53)

The effort of accommodation of the eye is very important in the play of working with a drawing of a three-dimensional figure and recognizing in it the figure, as well as regarding the capacity to shift from the figural to the conceptual and back. The use of technologies may be relevant with this respect, since

> Computer software for the teaching of 3D geometry should allow students to see a solid represented in several possible ways on the screen and to transform it, helping them to acquire and develop abilities of visualization in the context of 3D geometry. (Christou *et al*., 2007, pp. 3-4)

Our research interests in this paper are mainly on the question of seeing: what one does see when looking at a three-dimensional figure; if one is able to see a drawing as a three-dimensional figure, and its properties. At the same time, we are interested in understanding how the use of technology may help to face some of the problems that are related to seeing and the study of spatial geometry. Visibility concerns the first face of the coin, meant as making things visible through images, or "thinking in terms of images" as outlined by Calvino (1988). Visualization instead regards the second face of the coin, in which external means intervene in supporting making of images. Both faces are part of thinking processes, not a final result nor a static part.

In the literature, there are very few research studies about 3D geometry teaching and learning with technology. Some studies focused on students' exploration of the relationships between geometric figures in solid geometry (Accascina and Rogora, 2006; Baki, Kösa, and Karakuş 2008; Oldknow and Tetlow 2008), on students' perceptions (Bakó 2003), or on the use of virtual reality microworlds (Yeh and Nason 2004; Dalgarno and Lee 2010).

With the activity we consider here, we intend to open a debate about how the study of spatial geometry can be approached at high school, starting from known figures of the plane (quadrilaterals) and simple Euclidean properties of those figures that can be transferred in space through suitable definitions (holding for tetrahedra).

TEACHING EXPERIMENT AND ACTIVITY

The teaching experiment is part of a wider research, in which we became interested last year, with the main purpose to investigate *visual challenges* involved in doing spatial geometry and the role of technology to address these challenges (Ferrara & Mammana, submitted). To this aim, we started from the consideration of previous research that, given suitable definitions, a correspondence can be established between simple figures like quadrilaterals and tetrahedra (Mammana, Micale & Pennisi, 2009). The correspondence entails a movement from plane to space that we think of as a possible basis to introduce the study of 3D geometry at high school and that can be realised through the use of a Dynamic Geometry Software. The potentiality of the DGS is to allow for visibility and visualization of properties of the figures, bridging the gap between what can be seen and what can be learnt. The cognitive potential of the correspondence as a powerful didactical means is thus of interest for our research.

In particular, this paper regards a classroom activity that we carried out last May in a grade 11 class of a scientifically oriented high school in Giarre (Sicily). The students did not have any formal instruction about spatial geometry. Primary aim of the activity was to introduce them to the discovery that some properties that hold for quadrilaterals in the plane are preserved for tetrahedra in space, when the former are suitably defined. In this way, students are shown two different kinds of figures that share definitions and properties, and are stimulated not only processes of exploration in space but also the need for arguing the validity of certain properties. Here, attention will be drawn only to high school work.

Definitions and properties. To establish the correspondence, we gave the following definitions, using four non-collinear points and the six segments they identify:

1. A *convex Quadrilateral F* is determined by four coplanar points, A, B, C and D, any three of which are non-collinear, called *vertices*, and by the six segments determined by these vertices, AB, BC, CD, DA, AC and BD, called *edges*. The triangles identified by any three vertices are called *faces* of the quadrilateral.
2. A *convex Tetrahedron F* is determined by four non-coplanar points, A, B, C and D, called *vertices*, and by the six segments determined by these vertices, AB, BC,

CD, DA, AC and BD, called *edges*. The triangles identified by any three vertices are called *faces* of the tetrahedron.

Both figures have four vertices, six edges and four faces (letter *F* is expressly used in both cases, as well as same name for corresponding objects). No matter what *F* is, the couples of opposite edges and opposite face and vertex are equally defined. In addition, the segment joining the midpoints of two opposite edges is a *bimedian* of *F*, the segment joining one vertex with the centroid of the opposite face is a *median* of *F*. Given these definitions, the properties below are satisfied for *F*, again irrespective of its being a quadrilateral or a tetrahedron:

Property A. i) The three bimedians of *F* all pass through one point (*centroid*).

ii) The centroid of *F* bisects each bimedian.

Property B. i) The four medians of *F* meet in its centroid.

ii) The centroid of *F* divides each median in the ratio 1:3, the longer segment being on the side of the vertex of *F*.

The activities in which our students participated essentially focused on the discovery of properties A and B, and on their preservation in the passage from the quadrilateral to the tetrahedron. From the exploration of the quadrilateral in the plane, one can think of *moving* one vertex off the plane to transform the initial figure and to obtain a polyhedron, that is, a tetrahedron. So, the passage from plane to space can occur by means of the movement of a point, even encouraging to see changes and invariants.

Tasks and methodology. The whole experiment was based on the use of a DGS, Cabri Géomètre. We chose to use Cabri II Plus for the tasks about quadrilaterals and Cabri 3D for those regarding tetrahedra. Even if the students never met it before, Cabri 3D is the only DGS for 3D geometry released up to date. Providing learners with opportunities to redefine points, it furnishes a means to realize the movement from quadrilateral to tetrahedron: the *Redefinition* tool. One way this tool works is to change a point into a free point in space, requiring an action that involves the uppercase key of the keyboard (entailing a certain awareness of the action).

The activities were carried out in a computer laboratory, in which the students were divided into groups (of three/four people). Each group had two computer at disposal, to use Cabri II Plus on the one side and, on the other, Cabri 3D. The group work was alternated with class discussions guided by one of the authors, while the other author filmed one group and the collective moments. The teacher of the class was present as an observer of the activity of the students.

The tasks given to the students are shown in what follows, every voice corresponding to an assigned worksheet:

Q1. Definition and identification of the quadrilateral *Q* and of its main elements: vertices, edges, opposite edges, opposite face and vertex.

Q2. Definition and identification of the bimedians of *Q*; their properties.

Q3. Definition and identification of the medians of Q; their properties.

T1. Construction of the tetrahedron *T*; definition and identification of its main elements: vertices, edges, opposite edges, opposite face and vertex.

T2. Definition and identification of the bimedians of *T*; their properties.

T3. Definition and identification of the medians of *T*; their properties.

The worksheets on quadrilaterals had parallel worksheets on tetrahedra. In general, the tasks *Qx* presented three sections: Definition, Construction and Exploration, while those of the kind *Tx* contained sections about Definition and Exploration, except for the first task that asked first to construct the tetrahedron using the Redefinition tool.

The activities centred on conjecturing and discovering, being that of Exploration their main section. Formal proofs were not involved, even if reflections on ways to test conjectures with the DGS were made by the students for the properties on bimedians and medians.

ANALYSIS AND DISCUSSION

In this section, we present short episodes from the activity in Giarre that show the interplay between seeing and the use of the DGS. Attention is drawn to the way seeing in space is encouraged by the use of the software. In particular, we discuss how the discovery of properties in space is prompted and determined by processes of both seeing in space and knowing what happens in the plane. The role of the DGS is crucial in this phase, which leaves room for proof of the validity of properties.

The episodes involve three girls, Chiara, Cristina and Gabriella, that work together in front of two computers, one with Cabri II Plus and the other with Cabri 3D. We are at the second meeting with the class. In the previous meeting, the students have used Cabri II Plus to define the main elements of a convex quadrilateral *F* in the plane and to construct them, completing the requests of the first two worksheets, essentially bound to identifying the defined elements (*1Q*) and to exploring what happens with the bimedians when dragging the quadrilateral from its vertices (*2Q*). They have also learnt the concept of medians of a quadrilateral and how to construct them. During the second meeting, the students first summarised their discoveries in a class discussion ("*the bimedians all pass through one point*", "*we conjectured that the intersection point of the bimedians is the middle point of each bimedian*", from written texts). Then, they started the group work again, looking for properties of the medians of the quadrilateral (*3Q*), finding that they all pass through the centroid. After that, the class comes to deal for the first time with Cabri 3D and the passage from plane to space through the use of the tool *Redefiniton* (*1T*). The groups are first asked to construct a quadrilateral *F* in the visible (grey) part of the base plane, given by default by the DGS (Figure 1a). Then they have to move one vertex of *F* off the plane by redefining it as a point in space and to reflect on the new figure *T* (Figure 1b shows the arrows active during redefinition of vertex D).

First episode (May 21, *1T*). Chiara, Cristina and Gabriella are rotating the figure they obtained with the redefinition and they are watching how it changes. The task asks them to identify its elements, given their definitions:

Chiara:	There are four vertices A, B, C, D and four edges, that is, DC, BC (looking at the computer screen)…
Gabriella:	One, two, three, four, five, six (counting the edges and pointing to them on the screen), yeah
Chiara:	The four faces, one, two, three, four (pointing to the faces on the screen)… are (reading the worksheet) the vertices, the edges and the faces of *T*
Gabriella:	Here it is, the face opposite to this vertex (pointing to one vertex) is this one (tracing the corresponding triangular face on the screen; Figure 1c)
Chiara:	(suddenly, covering Gabriella's voice while she is still acting on the screen) Yeah! The faces, the faces of a pyramid! Ahhh, they became the faces of a pyramid! (aloud and astonished) Wow, wonderful! (satisfied; Figure 1c). Before they were the faces of a quadrilateral, then with this [meaning the Redefinition] they became those of the pyramid. Good! (convinced)

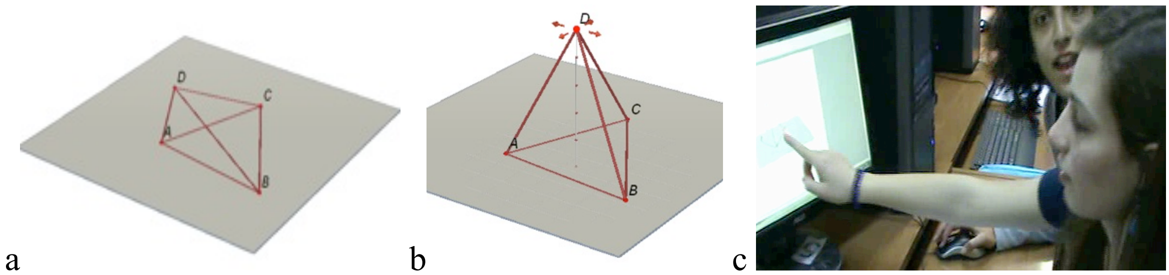


Figure 1. a) and b) Quadrilateral and redefinition of vertex D; c) Chiara astonished, and Gabriella acting on the screen

This brief episode shows the surprise of the students when they recognize and are able to see the faces of the pyramid/tetrahedron as previous faces of the quadrilateral (see Chiara's words "Wow, Wonderful!" and her facial expression in Figure 1c). The recognition occurs through the rotational movement enabled by the DGS that sustains the process of counting and identifying edges and faces of the tetrahedron. By means of this process the students start to discover the correspondence between the two figures ("Before they were the faces of a quadrilateral, then with this they became those of the pyramid"). The use of the verb "to become" is significant: it gives the idea that the passage from plane to space is occurred, as even the final use of the word "Good" accompanied by a convinced face points out. This action of becoming hides the change entailed by the redefinition of point D from a point of the base plane to a point off the plane.

Second episode (May 21, *2T*). The groups are requested to explore and conjecture what happens for the bimedians of a tetrahedron, again using the Redefinition (*2T*). To do so, they are given a Cabri 3D file that already contains a quadrilateral on the base plane together with its bimedians. Chiara, Cristina and Gabriella are looking at the quadrilateral on the screen, when Cristina begins to trace the three bimedians with her fingers.

Chiara: (taking the mouse) Do we have to rise point D? (trying to redefine it)

(The observer suggests rotating the figure to see if the point lies on or off the plane)

Chiara: (realizing that the point is on the plane) Ops, it’s on the plane! So, we didn’t redefine it (Figure 2a)

Observer: You have to get the arrows to be able to redefine it

(Gabriella takes the mouse and redefines the point correctly)

Chiara: Here it is, now it’s ok

Gabriella: Ok (going on to rotate the figure)

Chiara: Arrange it (to see it better). Further up, turn, there... What does it happen to bimedians? (taking the mouse and starting again to rotate the figure very quickly) Hmmm, that they meet in a point in space (Figure 2b)

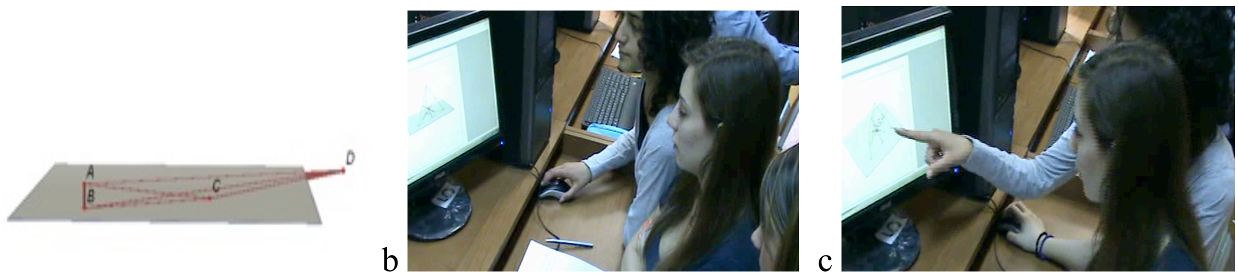
Gabriella: The same happening for a simple quadrilateral!

Chiara: That the middle point… Wait

Gabriella: That the middle point of the…

Chiara: These are opposite edges (pointing to them on the screen; Figure 2c), so it’s the middle point of the opposite edges

Gabriella: No, the point of intersection is the middle point of each bimedian, because the bimedian is the segment that passes through the two middle points of two opposite edges



a b c

Figure 2. a) Point D on the base plane; b) and c) seeing the meeting point of bimedians

As soon as Redefinition is applied in the right manner and vertex *D* is raised off the base plane, the three girls do not have troubles to realise that the bimedians “meet in a point in space” (Figure 2). The rotational movement given by the DGS again supports seeing, revealing this property. Through it, the students are constantly changing their visual perspective just as if they were looking at the figure from many different points of view, physically rotating around it. Gabriella, in a clever way, underlines the preservation of the property in space, evidencing the correspondence between the quadrilateral and the tetrahedron ("the same happening for a simple quadrilateral!"). This also entails the final conjecture that "the point of intersection is the middle point of each bimedian". The initial explanation given by Gabriella, concerned with the definition of bimedian ("the segment that passes through the two middle points of two opposite edges"), is not sufficient. The conjecture is instead verified using the distance tools of the DGS to measure suitable segments and to compare their lengths.

Once more, the students are able to reason in space transferring what they already know from the previous situation in the plane. Looking at the written texts produced by the groups, we can say that this is not an isolate case. For example, searching for properties satisfied by the medians of a tetrahedron, some of the students write: "Like for the medians of the quadrilateral, the point where they meet divides the medians so that their ratio is 3" (an expression not exactly correct, but that gives the idea).

The role of the DGS is fundamental in this activity. Its visualization capacities and the tool for redefining points allow to discover invariants and changes in the passage from plane to space, that is, from the quadrilateral to the tetrahedron. In addition, towards the end of the activities, we had the feeling that the students were keen on shifting in a natural manner from plane to space (what we may interpret as *to close their eyes and see*). This is not only due to an increased familiarity with the DGS, because their way to see in space is effectively changed and has been refined, so that they are able to treat the figure at point 3 in the introduction both as quadrilateral and tetrahedron, that is, to "see" in a single drawing two different geometrical figures. We may say to imagine one or the other, according to what they want to see for the purpose of the task. In this sense, our students learned things that are crucial in thinking about spatial geometry. They especially learned to accommodate the eye to make it work as a more expert eye that is able to discern relevant elements of a figure, to change perspective, and to make visible features that at a first glance are not present.

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