PLANE GEOMETRY: DIAGNOSTICS AND INDIVIDUAL SUPPORT OF CHILDREN THROUGH GUIDED INTERVIEWS – A PRELIMINARY STUDY ON THE CASE OF LINE SYMMETRY AND AXIAL REFLECTION

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Based on a detailed literature review, previous studies related to diagnosing and supporting (DS) children with mathematical learning disabilities are analysed with respect to the role of (plane) geometry. The results indicate that existing diagnosis and support instruments strongly focus on arithmetical topics while geometric contents are underrepresented or even non-existent. Then, first items DS line symmetry and axial reflection are implemented in 6th grades in the form of a test in order to specify children's understanding of these concepts and to find suitable items for the DS instrument. Finally, the test results are compared with findings through an individual interview conducted with a 5th grader beforehand.

Key words: diagnosis in geometry, axial reflection, line symmetry, figural concepts.

INTRODUCTION

Diagnosis and individual support has proved itself valuable to learn more about children's ways of mathematical thinking and to screen their misconceptions[1] and lacking mental models in order to contribute to a change for the better. A number of publications provide descriptions and lists of pupils' difficulties, error types and misconceptions, as well as suggestions how to foster particular mathematical issues. However, existing studies and commercial instruments for DS focus mainly on arithmetical issues. The field of geometry is either not being covered or only being minimally addressed. Consequently, it's even more important to develop a diagnostic instrument to assess children's geometric misconceptions and fields of difficulties in order to gain support points for the individual fostering. Thus, it is our aim to develop item-based interview guidelines suitable for detecting children's misconceptions and solution strategies regarding geometric issues and problems as well as to provide basic individual support rudiments. Our first investigations focus on line symmetry and axial reflection since there are a number of studies that describe pupils' misconceptions and error types (Küchemann, 1993; Xistouri, 2007; Bell, 1993) and are conductive to the development of DS items and interview guidelines. The purpose of this paper is to present the first outcomes of our study concerning line symmetry and axial reflection. It presents the items implemented in 6th grades in the form of a test and its analysis results regarding children's understanding and misconceptions of axial reflection. The overall aim is to develop task-based interview guidelines for the diagnosis and support of lower secondary students' figural concepts of line symmetry and axial reflection. This process will draw upon the theory of figural concepts by Fischbein.

THEORETICAL FRAMEWORK

The Concept of Line Symmetry and Axial Reflection

Hoyles & Healy (1997) investigated the processes through which pupils come to negotiate mathematical meanings for reflective symmetry by describing a micro-world, Turtle Mirrors. It is used "to help students focus simultaneously on actions, visual relationships and symbolic representations regarding reflective symmetry" (Panaoura, Elia, Stamboulides & Spyrou, 2009, p. 46). They also describe students' primitive and intuitional variety of strategies for solving paper and pencil tasks on reflective symmetry. The reflection of objects in horizontal or vertical axis was easier for the students than in slanted axis. Hoyles & Healy (1997) describe how pupils use an approximate strategy derived from paper folding while reflecting in slanted axis – called 'the strategy of imagining a vertical axis (IVA)' in this paper.

Pupils face different problems when constructing plane reflections in a line and identifying the lines of symmetry in plane figures. Schultz (1978), Grenier (1985) and Küchemann (1993) identified the following factors as relevant: (a) direction of the axis (vertical, horizontal, slanted (45°), other); (b) complexity of the object being reflected; (c) presence or absence of a grid; (d) slope of the object and (e) size of the objects and distance from the axis of reflection. The first four were incorporated by Küchemann into a structured sequence of questions and the levels of response were identified as global, semi-analytic, analytic and analytic-synthetic. In *global* responses the object is considered and reflected as a whole with no reference to particular parts, angles, or distances. In *semi-analytic* responses, a part of the object is reflected first and the rest drawn from its matching the original shape and size. In *fully analytic* responses, the object is reduced to key-points, each reflected individually. These are connected and the result is accepted even though sometimes the image looks wrong. In analytic-synthetic responses, the global and analytic responses are coordinated so that the image is precise and also looks correct (Bell, 1993, p. 130). Küchemann indicated that students have some informal understanding of geometric transformations such as reflection and rotation. However, children experience difficulties when working on shapes which involve these transformations.

Bell's study (1993) consisted of interviews and a diagnostic teaching experiment with students aged 11-12 years. This, too, revealed misconceptions. We take one example: children believe that horizontal/vertical objects have horizontal/vertical images or that horizontal objects have vertical images and vice versa.

There are several studies which deal with students' conceptions of reflective symmetry and their difficulties in understanding the concept; however, there is more to be uncovered and explored. As an example, "[l]imited attention has been given to interrelations among students' concept image of reflective symmetry and the use of different representations of the mathematical concept" (Panaoura, Elia, Stamboulides & Spyrou, 2009, pp. 47-48).

The Theory of Figural Concepts

Fischbein (1993) introduced the notion of *figural concepts*. He made an attempt to interpret geometrical figures as mental entities possessing simultaneously both conceptual and figural properties. According to the theory of figural concepts, the main objective is the development of the interaction between the figural and the conceptual aspect. Fischbein (1993, p. 160) stated that

> [a]lthough a figural concept consists of a unitary entity (a concept expressed figurally) it potentially remains under the double and sometimes contradictory influence of the two systems to which it may be related – the conceptual and the figural one. Ideally, it is the conceptual system which should absolutely control the meanings, the relationships and the properties of the figure.

Various students' difficulties in geometrical reasoning can be interpreted in terms of such a rupture between figural and conceptual aspects of figural concepts – possibly even their difficulties in answering questions referring to axial reflection. The theory of figural concepts provides a powerful tool and offers a theoretical framework for our analysis of students' understanding of axial reflection and other geometric topics.

METHODOLOGY

The purpose of this study is to compile task-based interview guidelines for the diagnosis and support of lower secondary students' understanding of axial reflection, and thus for the development of the interaction between their figural and conceptual aspects of axial reflection. To achieve this, a pre-test was conducted and evaluated with Küchemann's methodology (Küchemann, 1993). Its results, that is, the solution rate of the items and emerging students' typical misconceptions while working with those items, will be examined in order to select suitable interview-items. The subjects were 195 6^{th} graders who were chosen from nine classes and three schools. The students were given two invented tasks on line symmetry (including three items) and four tasks on axial reflection (including 13 items). The latter were adapted from the CSMS transformation geometry test restricted to an investigation of reflection and rotation (Küchemann, 1993). However, in order to focus on basic knowledge and save working time, both difficult items (e.g., object intersects the axis) and equivalent items of the CSMS transformation geometry test were excluded from the pre-test.

The items for line symmetry will give an overview about students' concepts of symmetry. The first task pertained to their associations with line symmetry and axial reflection; the first item of the second task asked the students to draw images with one, two and no axis of symmetry. The second task provided eight flags of European countries for which the children should decide by marking with a cross if the given flag is symmetric or not, state and draw the right number of axis of symmetry.

The 13 items for axial reflection (tasks 3 to 6) included working on squared and blank paper with and without the use of the set square. Items of tasks 3 and 6 involved the drawing of the image. Task 4 (see fig.1) requires sketching the axis of reflection between pairs of figures (4.1) or stating that and why this was not possible (4.2). Task 5 provided several points as possible reflections of point A. The students had to choose which of the given points was the image of A and explain their choice (see fig.1). Explanations encourage students to think analytically about the properties of reflection, even if they had originally made an intuitive choice.

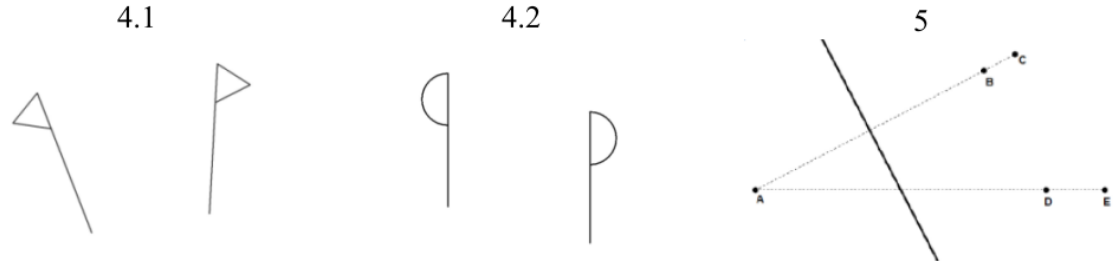

Figure 1: Items to tasks 4 and 5

In task 6 (see fig.2) students were allowed to use the set square; the remaining items required working without the use of the set square and other tools as well. In comparison, the equipment needed for the CSMS transformation geometry test was a ruler marked in centimetres; protractors and set squares should not be used for that test. The reason for our method lay in the fact that an individual interview with a 5th grader has shown that working with the set square provides wide support for the students. Many students have tendency to rely on procedural aspects of axial reflection, which is, e.g. positioning the set square in the taught way, measuring the distance of the object to the axis and finally transferring it to the distance of the image and the axis. This also became evident in the students' responses, especially in task 5. Since the students weren't allowed to use any tool for the test items – task 6 apart – and many of the questions involved drawing, regions had to be defined to delimit what was to be regarded as correct. For this, we used the marking scheme of Küchemann and the printed acetate sheets which he provides. He, too, defined regions by devising rules that took into account such factors as distance from the axis, the slope of the axis and whether or not a ruler was to be used. Thus, five codes were allocated to children's responses according to Küchemann's marking scheme. Code 1 includes responses which are correct and Code 2 those that are adequate, that is, not precise enough to be regarded as correct but also not obviously wrong. Code 3 refers to overt errors, e.g. reflecting horizontally or vertically when the axis is slanting or drawing the image parallel to the object. Code 4 refers to all other responses and Code 5 to unworked items. Two raters were involved in marking the students's responses and in assigning Codes to them. In order to perform reliability testing the Cohen's kappa coefficient was calculated. With $\kappa \approx 0{,}84$, it's safe to assume that a good inter-rater-agreement is present in this case.

Küchemann (1993) classified his reflection items into two types instead trying to discuss the effect on children's performance of the various combinations of the above mentioned features (a)-(d). These two types differ, generally, in terms of the strategies needed to produce Code 1 and Code 2 responses. Type A covers items where the axis is vertical (or horizontal) or the object is a single point (3.1, 3.2, 3.4, 3.5, 3.6). Type B includes items where the axis is slanting and the object is a line or a flag (3.3, 3.7, 6.1, 6.2) (see fig.2). So, type A items can be regarded as involving only one slope (of the object *or* the axis) because, having a vertical axis, the ability to reflect in a direction perpendicular to the axis requires merely the knowledge that the reflection takes the object to the other side. Type B items involve both the slope of the object *and* the axis.

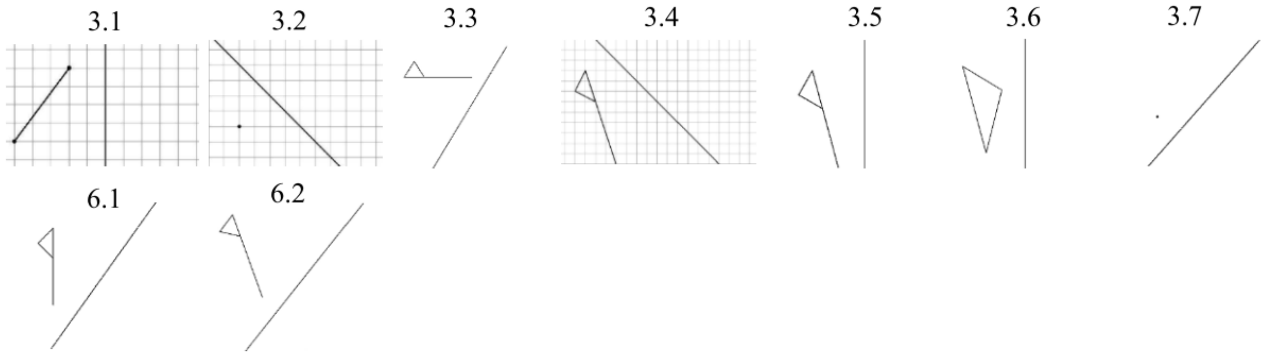

Figure 2: Items to tasks 3 and 6

Referring to the 13 items for axial reflection, each response to an item in each of the three reflection tasks was assigned one of the five codes. The percentages of success (Code 1) were calculated for each item. The items were grouped according to the percentages of success and item characteristics. The percentages were used inter alia for comparing the difficulties students experienced, both in Küchemann's test and in our sample, while working with the items, that is, for sorting the items by levels of difficulty. This is important because interviewed students will be provided with items also sorted by levels of difficulty. In order to intervene as a qualified professional, the challenging function of the interviewer will be to find out as much as possible about the student's figural concept of axial reflection or rather about the conflict between the conceptual and figural aspects generated by the given items. For this purpose, students' main strategies or error types in each item are worked out, which is useful for the interviewer to be able to shed light on critical issues while interviewing the child.

RESULTS

The following tables 2 and 3 present the percentages of students' success in solving the reflection items. Since the items are grouped according to the percentages of success (Code 1), these tables also illustrate with which items students could cope more easily. The gradations in the tables are correlated to the item-levels of Küchemann (1993) with a factor of 0,90. A t-Test ($p>0.86$) shows that there is no statistical difference among Küchemann's results of 1980 and ours of 2012. This implies, considering our sample, that there is no change in students' misconceptions and understandings of axial reflection since 1980.

The solution rates of task 4 reveal that students coped more easily with stating that it is not possible to draw a mirror-line between pairs of figures (4.2) then with finding the mirror-line when it was possible (4.1) (see fig.1). A percentage of 66% of the students stated correctly in 4.2 that a mirror-line is non-existent. As for the results of table 1, 51% justified this statement by saying that the two figures are shifted or don't have the same height and 11% by stating that both figures are parallel or don't face each other; 28% of the students drew vertical or diagonal axis between the two figures, which is again similar to Küchemann's results where 26% did so (Küchemann, 1993).

TASK	PERCENTAGE OF STUDENTS' (IN)CORECT (–/+) & NO (/) RESPONSES			PERCENTAGE OF GEOMETRIC NOTIONS IN STUDENTS‘ RESPONSES				
	+	-	/	A is faced by B/D	distance	slanted axis	direction	⊥
5	78	15	7	13	52	15	9	11
				shifted	face-	parallel-	other or no responses	
4.2	66	28	6	51	5	6	38	
4.1	53	42	5	no reasoning demanded				

Table 1: Items grouped according to the % of success & item characteristic 'reasoning'

In task 5 (see fig.1), 78% of the children chose B as the image of the point A whereas 9% chose D. 52% of the students gave explanations that referred merely to the distance and 9% merely to the direction. However, only 13% of the responses referred to both the distance *and* the direction of the point in relation to the axis. Among those children who focused on only one property, 13% justified their choice with 'A is faced by B(/D)', 15% chose B 'because the axis is slanted' and only 11% used the terms like 'perpendicular', 'right angle' or '90°'. 73% of those who referred to perpendicularity also mentioned distance in their explanation. Those who chose point D explained their choice by referring to distance and/or by stating that, e.g. 'D is directly opposite to A" or "A is faced by D". In comparison, in the CSMS transformation geometry test, 21% gave explanations that referred to the distance *and* the direction, 33% focused merely on the distance and 20% on the direction (Küchemann, 1993, p. 141).

Considering the students' associations with the two topics in task 1, their tendency to rely on procedural aspects of reflective symmetry becomes obvious. 42% associated with axial reflection and line symmetry tools, with 24% mentioning the set square, 6% the ruler, 4% the compass and 8% the pencil. The terms 'right angle' and 'perpendicular' occurred only in 5% of the responses and 'sameness' in 9%. It is obvious that the properties are slightly associated with symmetry and reflection tools. Especially the set square (see fig.4), which assists students in solving symmetry and reflection items, seem to play a more prominent role in the students' associations than the conceptual aspects. Regarding students' explanations in task 5, the aspect of distance seems to be more internalized than that of perpendicularity. One root cause may possibly be the broad use of the set square whereby students just have to measure the distances to the axis; the aspect of perpendicularity is hidden behind the center line of the set square (like in a black box) which is to be positioned onto the axis in order to draw perpendicular lines. Thus, many students do not realize which geometrical action is hidden behind this 'positioning'.

The supporting role of the set square is also striking in a videotaped interview with a 5th grader who is struggling with drawing the image of a 'modified' triangle in a slanting axis without the support of any tool. After a couple of minutes the child produces the reflection of the given object –nevertheless the result is the image of a reflection in a vertical axis because the child ignored the slope of the axis and imagined a vertical axis (IVA) (see fig.3).

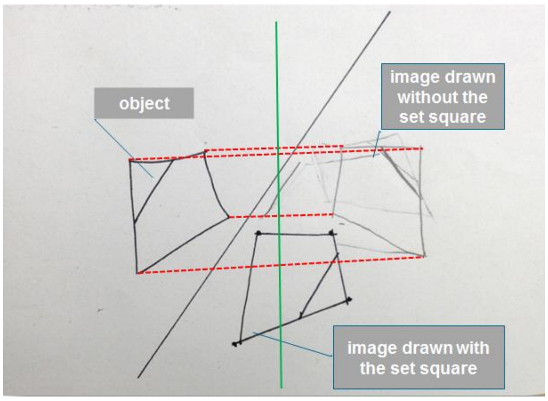

Figure 3: Students' strategy of imagining a vertical axis (IVA)

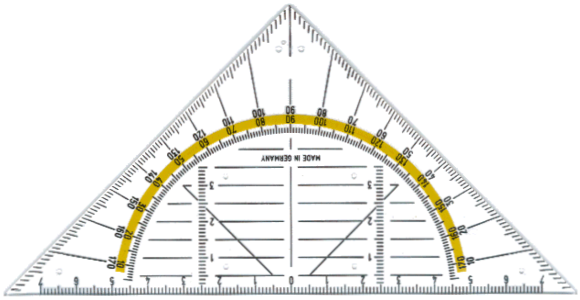

Figure 4: The set square

The child, still not aware of the answer's incorrectness, is asked by the interviewer to check her result with the set square. Within a few minutes the child produces the precise correct answer on the same sheet by relying only on procedural aspects. In this case study, the support of the set square is enormous and it is our future aim to investigate other cases concerning this matter through guided interviews. Does the set square inhibit or delay the development of the interaction between students' figural and conceptual aspects of axial reflection by providing too much assistance? Is the set square a stumbling block for the internalisation of axial reflection?

The results of tasks 3 and 6 depict students' typical misconceptions. 3.1, 3.2 and 3.6 are items where the axis is vertical or the object is a single point. It is noticeable that 8% of the pupils who could cope with 3.1 couldn't do so with 3.2. Items 3.1 and 3.2 caused least difficulties for the students. In the former, 2% of the students' responses were assigned to Code 3, including 67% reflecting horizontally (RH) and 22% drawing the image parallel to the object (IPO). In the latter, 7% of the students' responses were allocated to Code 3, with all 7% RH. For the items 3.2 and 3.6 the percentage making overt errors (e.g. RH) was virtually the same. 8% of the responses to item 3.6 were Code 3 responses including 78% RH (see table 2). The presence of a grid in 3.2 seems not to be a simplification for the students. The presence of a slanting axis in both 3.2 and 3.6 and of a slanting object in 3.1 seems to cause RH.

ITEM	PERCENTAGE OF CODE					PERCENTAGE OF ERROR TYPES (CODE 3&4)							
	1	2	3	4	5	RH/ RV	IPO	IVA	D-	S-	T	ROT	Others
3.1	91	7	2	0	0	67	22	0	0	0	0	0	11
3.2	86	4	7	2	1	78	0	0	0	0	0	0	22
3.6	63	19	8	6	4	78	0	0	11	0	0	0	11
3.4	61	0	0	36	3	0	22	0	50	3	5	3	17
3.5	55	0	0	41	4	0	0	0	54	29	7	0	10

Table 2: Items grouped according to % of success & item characteristic 'type A'

Responses to items 3.4 and 3.5 were assigned to Code 1, 4 and 5 (Küchemann, 1993). It is striking that in 3.4 50% of the error types related to the distance (D-). 22% drew the image parallel to the object (IPO), 3% drew the wrong size of the object (S-), 5% translated the object (T) and 3% confused reflection with rotation (ROT). In 3.5 54% reflected failing the distance and 29% the size of the object. Both items involved a vertical axis and a slanting object – in 3.4 a flag and in 3.5 a triangle. Here, the complexity of the objects had effect on children's performance.

3.7 and 6.1 are items involving a slanting mirror-line whereas the object to be reflected is horizontal or vertical. Working with items 3.7 and 6.1, the students experienced most difficulties. Virtually the same percentage of children made overt errors (e.g. IPO, RH/RV) despite the fact that the students were allowed to use the set square in 6.1. Code 3 and 4 responses to item 3.7 involved 31% IPO and 21% IVA; 10% translated the object and 17% confused reflection with rotation (see table 3). The percentage of students using the strategy IVA was even greater in 6.1: 47% imagined a vertical axis and reflected then. 18% drew the image parallel to the object and 20% rotated the object.

ITEM	PERCENTAGE OF CODE					PERCENTAGE OF ERROR TYPES (CODE 3&4)							
	1	2	3	4	5	RH/ RV	IPO	IVA	IPA	O-	T	ROT	Others
3.3	45	18	20	8	8	18	26	7	12	14	5	0	18
6.2	40	29	15	4	13	19	15	8	0	0	0	35	23
6.1	39	20	35	2	4	0	18	47	0	0	0	20	15
3.7	25	29	32	4	10	13	31	21	0	0	10	17	8

Table 3: Items grouped according to % of success & item characteristic 'type B'

Items 3.3 and 6.2 involve two slopes (object and axis), which implies that the slope of the object has to be related to the axis. Many children seemed to find the coordination that is required difficult and either ignored the slope of the axis entirely or ignored its effect on the slope of the image or didn't work on the items. In 26% of the Code 3 and 4 responses to item 3.3 the image was drawn parallel to the object and in 18% the object was reflected horizontally. In 14% students confused the orientation (O-) and in 12% they draw the object parallel to the axis (IPA). Code 3 and 4 responses to item 6.2 included 19% RH and 8% IVA, 15% IPO and 35% ROT. In both 3.3 and 6.2, the percentage of IVA errors was low. The proportion of children answering both items correctly is virtually the same, also the proportion making overt errors even though in 6.2 the use of the set square was allowed. Here, the question which arises is why there is a low solution rate in 6.1 and 6.2 in spite of the fact that the set square was allowed. Did the work without the use of a tool in the first 11 items generate a conflict between the figural and conceptual aspects of students' figural concepts of axial reflection? And is this related to the fact that the students were confused when they had to use the set square in task 6? Additionally, about 45% of the students' total responses in the pre-test involved IVA more than once. However, there's no perceptible pattern since the majority didn't make this mistake more than twice. Even so, 45% of the students applied IVA systematically.

CONCLUSION AND DISCUSSION

Appropriate test-items and the pre-test's results will be used not only to compile item-based interview guidelines aiming to *diagnose* lower secondary students' understanding of axial reflection, that is, to develop their figural concepts of axial reflection, but also to *support* children to revise them. Certainly, one has to consider that harmonizing the two components of figural concepts is neither spontaneous nor simple (Mariotti, 1995). Hence, in a next step, individual interviews will be conducted with 6th graders in order to help students to develop the interaction between the figural and conceptual aspects of axial reflection. An intervention or an attempt to do so could be made by generating a conflict between the two aspects. Destabilizing repeatedly a student's figural concepts and making him/her adapt conceptual aspects can possibly have positive effects in the development of *his or her* figural concept of axial reflection.

Based on the above analysis and results, several questions have been raised and need to be answered through further analysis and investigation. Future interviews may clarify the results of the quantitative part of this study. Among other issues, we will focus on questions regarding the role of the set square in the process of learning and understanding axial reflection as well as regarding the development of students' figural concepts of axial reflection: firstly, does the set square inhibit or delay the development of the interaction between students' figural and conceptual aspects of axial reflection by providing too much assistance? Secondly, is the set square a stumbling block for the internalisation of axial reflection? Lastly, did the work without the use of a tool in the first 11 items generate a conflict between the figural and conceptual aspects of students' figural concepts of axial reflection? At this point, the study of Son (2006) must be considered. It reveals that a large portion of pre-service teachers has misconception and limited understanding of reflective symmetry. Furthermore, "they have tendency to rely on procedural aspects of reflective symmetry when helping a student understand reflective symmetry correctly although they recognized a student's misconceptions in terms of conceptual aspects" (Panaoura, Elia, Stamboulides & Spyrou, 2009, p. 47). Therefore, it is important that we attempt to answer the above questions since this has implications not only for the students, but also for teacher educators and teachers.

NOTES

1. In this paper the word "misconception" has been used instead of "conception". This is because the German notion *Schülerfehlvorstellungen* (= students' misconceptions) is common in the field of diagnostics in Germany. That is why we followed the perspective of (Son, 2006) and (Bell, 1993).

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