STUDENTS' MODELLING OF LINEAR FUNCTIONS: HOW GEOGEBRA STIMULATES A GEOMETRICAL APPROACH

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The use of technology has crucial influences on mathematical modelling. We present a study with a class of 8th graders solving tasks involving linear models, where students regularly used GeoGebra. Our aim is to examine students' approaches to contextualized problems in the "technology world" of the modelling activity. The results show that students skipped much of the algebra work, and rather chose a geometrical approach, harnessing the affordances of the tool.

INTRODUCTION

The impact of digital tools in mathematics teaching and learning is, to a great extent, concerned with the changes that the use of technology operates in the forms of understanding and approaching mathematical ideas and processes. In many ways technology challenges the traditional hierarchy and disconnection of mathematical topics and it also reshapes the nature and purpose of mathematical representations in doing mathematics. This has clear implications on a learning context based on developing mathematical models of real situations given "the way technologies can provide multiple connections in mathematics, supporting a student's holistic development of mathematical understanding" (Pead, Ralph & Muller, 2007, p. 318).

Developing and exploring mathematical models with technological tools reveal new sides that go far beyond the idea of gaining more computational or graphical power in dealing with mathematical models. In fact, as some researchers have been suggesting, the well-known modelling cycle needs to be re-conceptualised to integrate a third world – the technology world (Greefrath, 2011; Greefrath, Siller & Weitendorf, 2011; Siller & Greefrath, 2010). Not only the modelling cycle can be augmented to include a third world where the computer model and the computer results are fundamental parts but, most importantly, the impact of digital tools occurs at all stages of the modelling cycle. The formulation of the mathematical model and the computer model are fused together and the same goes for the application of the mathematical model and the implementation of the computer model.

As Greefrath (2011) points out, one of the consequences of using digital tools is the "algebracising" of numerical data. In fact, different available software and computer packages can provide algebraic representations of real data inputs, offer a graphical representation of the generated algebra, and additionally allow establishing connections between them dynamically.

In the case of real problem situations involving linear change, the mathematical models involved are typically associated with the concept of linear functions, including its algebraic formulation, together with its tabular and graphical representation. However, when using GeoGebra students may easily get an algebraic expression of a linear function just by plotting two points of the graph and choosing the tool to create a line through two points; the equation of the line appears as an independent object. Therefore the process of creating and applying a mathematical model may change significantly. The act of building a model is fused together with the computer outputs resulting from entering a table of numerical data and then translating it into other mathematical representations (a graph or an equation).

Given such significant consequences of computer use in students' access to different interconnected representations of linear variation, it is important to investigate how students' approaches to linear models are influenced by the affordances of GeoGebra.

This study focuses on a class of 8th graders developing mathematical and computer models with GeoGebra, and aims to investigate how students' ways of formulating and applying linear models are shaped by the technological tool.

THEORETICAL FRAMEWORK

The emphasis given to particular goals behind mathematical modelling in education has enabled researchers to distinguish among different perspectives (Kaiser & Sriraman, 2006). Accordingly, both a conceptual modelling and a contextual modelling perspective are important roots to situate the theoretical stance of this study. Realistic situations (some of them requiring experimental data collecting) are seen as providing the opportunity to elicit students' broad understanding of linear functions, like constant rate of change and parameters involved in linear graphs. Following that initial basis, contextual problems are seen as opportunities for conceptual development, connecting different aspects of linear models, like relating tables to graphs and to equations in finding answers to particular problem-based questions.

Different theoretical standpoints have been suggesting possible ways of looking at the effects of introducing digital tools in students' understanding of mathematical models. One way of addressing the interplay between modelling and technology focuses primarily on the medium that the modeller is using and stands on the idea of co-action, which acknowledges an interactive influence between the user and the technological tool (Moreno-Armella, Hegedus & Kaput, 2008).

> (…) we introduce the idea of co-action to mean, in the first place, that a user can guide and/or simultaneously be guided by a dynamic software environment (Moreno-Armella, Hegedus & Kaput, 2008, p. 102).
>
> The student and the medium re-act to each other and the iteration of this process is what we call co-action between the student and the medium (Moreno-Armella & Hegedus, 2009, p. 510).

From that point of view, the use of technological multi-representational tools frames students' representational choice (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel 2009) and this is reflected on their modeling approaches when solving modelling tasks.

Studies on the strategies developed by students in problem solving tasks have shown that they tend to avoid the algebraic treatment of the problem and prefer non-algebraic routes as those based on arithmetic reasoning, on trial and error, working backwards, etc., all summing up to a certain compulsion to calculate rather than to do algebra work (Stacey & MacGregor, 2000).

A similar movement of deferring algebraic approaches is reported by Yerushalmy (2000), where students made intensive use of technology in a function approach to school algebra, through modelling tasks. Based on linear break-even situated tasks proposed to pairs of students in three interviews, separated in time, the study shows how students' strategic approaches to the problems evolved: from graphical representations to predominantly numerical methods, to relations between quantities, to graphs of functions and finally to the algebraic expressions.

> The learning starts with graphical representations of variations, used later on to analyze patterns of numbers by watching the behaviour of the increments, moves on to analysis and construction of relations between quantities that are defined globally, to accurate graphs, and then to explicit expressions (Yerushalmi, 2000, p. 142).

As already stated the technology world brings in more opportunities for students to decide which representational modes they find the most efficient to formulate and apply mathematical models to real situations. Therefore students' approaches to mathematical modelling are likely to evolve within such versatile contexts and to match different levels of mathematization (from the real world to mathematics or within the mathematical world). In such evolving processes of representational decisions co-action between the student and the tool becomes an important concept in that the tool offers an answer (e.g. the algebraic expression) as a result of an alternative representational act from the user (e.g. plotting a graph).

Our study looks at the ways in which students approach linear models in a multi-representational environment (more precisely to see how they are formulated and applied in a set of tasks).

RESEARCH METHOD

The teaching experiment supporting this study was developed in a class of 8th graders aged 12-14, the majority being 13 years-old, from a public school located in the metropolitan area of Lisbon. The class is characterised by an overall good level of mathematics achievement. The teaching experiment was designed in line with new Portuguese curricular orientations, according to which understanding in algebra topics is to be supported by modelling real situations. A sequence of 7 tasks was developed over a period of one month and took seven lessons with different durations: 3 lessons of 90 minutes and 4 lessons of 45 minutes. All lessons took place in a computer lab where students were organised in pairs, each pair having one computer to work on. In each task students were to find a mathematical model for a problem situation by selecting information, interpreting the situation, creating a model and applying it to find answers to specific questions about the real context.

Some of the tasks required that students engaged in real data collection, either from an experiment performed in the class or outside. All the tasks were conceived to activate the use of mathematics, namely the concepts of linear variation and linear function, in connection to contextualised questions. The tasks evolved from problem situations aimed at eliciting linear models' general properties to more focused problems where the context was explored to stimulate the use of linear models for obtaining particular solutions.

Students had already some experience in using GeoGebra from the previous school year; they were familiar with the Graphics View, the Algebra View and the Spreadsheet View of the software. Students were not completely aware of the fact that once they created geometric constructions on the Graphics View the equations were displayed in the Algebra View, as a result of the interactive nature of graphics and algebra in GeoGebra. The class teacher decided to allow students to discover for themselves any additional details of the program while performing the tasks. Students were not compelled to use the computer but rather they were free to choose how to solve each task. Therefore, in the same task, some students could work with GeoGebra, others only with paper and pencil, and even others could use both.

For this study, different types of data were collected: the work produced by the students (written records, files created in GeoGebra with access to the construction protocol), the daily log of class observation, and information recorded on audio and video of two pairs of students.

A qualitative methodology and a case study design with strong descriptive and interpretative dimensions were adopted. From each of the pairs that were videotaped one student was elected to become the core of a case. Nevertheless, the other member of the group and also whole class discussions were sometimes integrated in the case report.

In the following we present a segment of the case of Pedro, one of the students who belonged to a videotaped group, which in a way reveals exemplar instances of the kind of modelling approaches that took place in the class when GeoGebra was used as a multi-representational tool.

DATA DESCRIPTION AND ANALYSIS

A selection of four tasks is presented for the case of Pedro and his partner, Diogo. The selected tasks (1, 4, 6, and 7 of the sequence) are intended to illustrate the development of students' approaches to modelling over time.

Task A: Water filling

In the class students performed a simple experiment of filling beakers with water from a tap and recording the volume of water after successive equal time intervals. The water flow was changed and the experiment repeated two or three times.

1. In filling the beakers with water what are the variables involved and what is the relationship between them?
2. Find an equation to describe each experiment. Describe how the water flow affects the equation.
3. What is the volume of water after 15 seconds, in each case?
4. How long does it take to fill up a volume of 5 liters, in each case?

Pedro and Diogo entered the collected data organised by columns in the Spreadsheet View and plotted the several points (pairs of time and volume) in the graphical area. Then they easily created several straight lines in the Graphical View until they found the one that they considered the best fit. The students were surprised to see that the equation of each line was immediately given in the Algebraic View of GeoGebra.

Pedro: Look, it also makes the equation!

To answer the question of finding the volume after 15 seconds, Pedro and his partner decided to enter the equation x=15 in the input bar and obtained the correspondent vertical line (equations are immediately displayed in the Algebra View and graphed in the Graphics View). Then they noticed the intersection points appearing in each of the previous graphs and saw the presented coordinates of those intersection points.

Teacher: What are you doing then?

Pedro: We are making the intersection between this and the other lines.

Teacher: Oh, very well thought out.

Pedro: And we have discovered the points of intersection (in the Algebraic View).

By reading the coordinates of the intersection points in the Algebra View, Pedro answered: Exp. 1 – 435 s; Exp. 2 – 141 s; Exp. 3 – 115 s.

Following the same reasoning, the two students entered the equation y=5000 (volume in ml) and determined the intersection points between the horizontal line and the linear models of each of the experiments. Thus they obtained the time (x-coordinate in the Algebra View) needed to fill the beaker with 5 litres, for each water flow.

Task B: The stack of shopping baskets

On a trip to the supermarket in their extra-school time, students conducted some data collection, measuring a stack of shopping baskets with a variable number of baskets. They measured the height a stack of one basket, two baskets, and so forth. As the number of baskets increased, students recorded the height of the stack on a table.

1. Create a graph that fits the data collected in your table.
2. Find a model to represent the height of a stack of shopping baskets depending on the number of baskets in the stack.
3. What is the height of a stack of 50 baskets? Explain.

In the class, using the information recorded in their tables, they started to create a graph in GeoGebra, showing no difficulties in performing this action. Pedro and his partner began by using the Spreadsheet View and entered the data organised in columns, then marked the points in the Graphics View and obtained a straight line. Instantly, in the Algebra View, they got the expression y=8x+30 (figure 1).

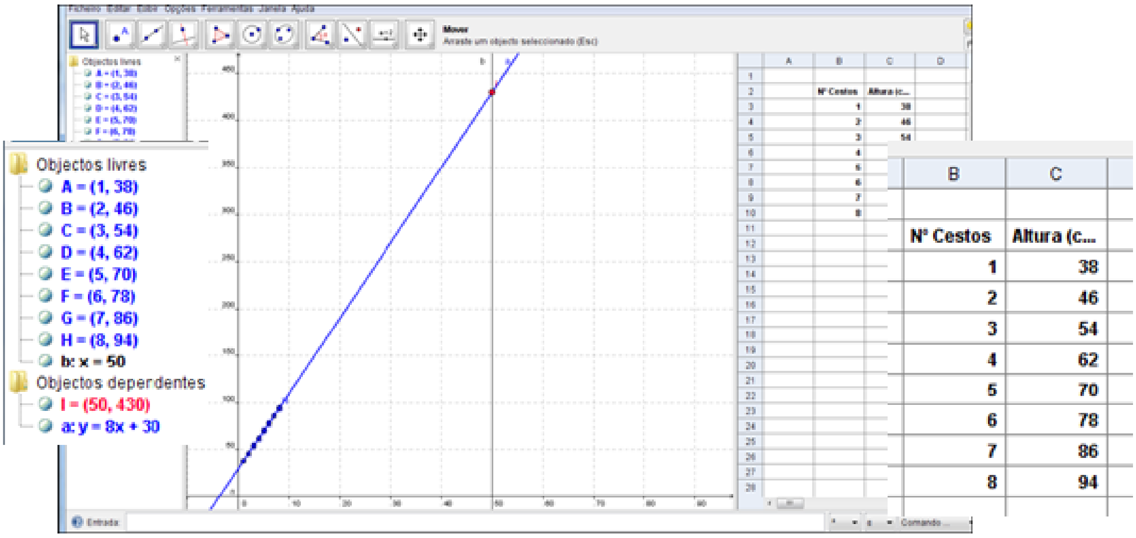

Figure 1: Computer model of the height of the stack (screenshot of students' file)

The question of finding the height of a stack of 50 baskets showed different approaches in the class. Two approaches were used among the several pairs of students. Some took the algebraic expression obtained in GeoGebra and used paper and pencil, assigning the value 50 to x to compute the y. Other pairs chose to work in GeoGebra, entered the equation of the vertical line and by intersecting lines obtained a point and its coordinates. This was the case of Pedro's group who arrived at point I(50,430), as shown in figure 1.

Pedro's explanation on the processes used was given in the following way:

Pedro: But we also found another way to solve the equation.

Teacher: How is that?

Pedro: Well, x was number of baskets, and as the number of baskets had to be 50, we made 8 times 50 plus 30. And we got 430.

Teacher: And besides, how did you do it the other way?

Pedro: It was like this: we draw the line x=50 and intersected it with the line already there and it gave the point where y was 430.

Task C: The cost of having your own car

The problem statement provided some data on the monthly cost of owning a car depending on the distance travelled. Students started by using the Spreadsheet View to enter the given data.

The cost of owning a car depends on the number of kilometres travelled per month. Based on the information published by "Time Magazine", the cost changes linearly with distance, and it is 336 € per month for a distance of 300 km and 510 € per month for a distance of 1500 km.

1. Find an equation that expresses cost versus distance.
2. Make an estimate of the monthly cost to a travelled distance of: 1000 km; 2000 km.
3. Find the maximum distance not to exceed a monthly cost of 600 euros. Explain.
4. Find the y-intercept in the graph of cost versus distance. How do you interpret it?

Then they plotted the two points from the table and draw the line through them in the Graphics View. Again GeoGebra allowed students to have the graph of the function and its algebraic expression simultaneously displayed (figure 2).

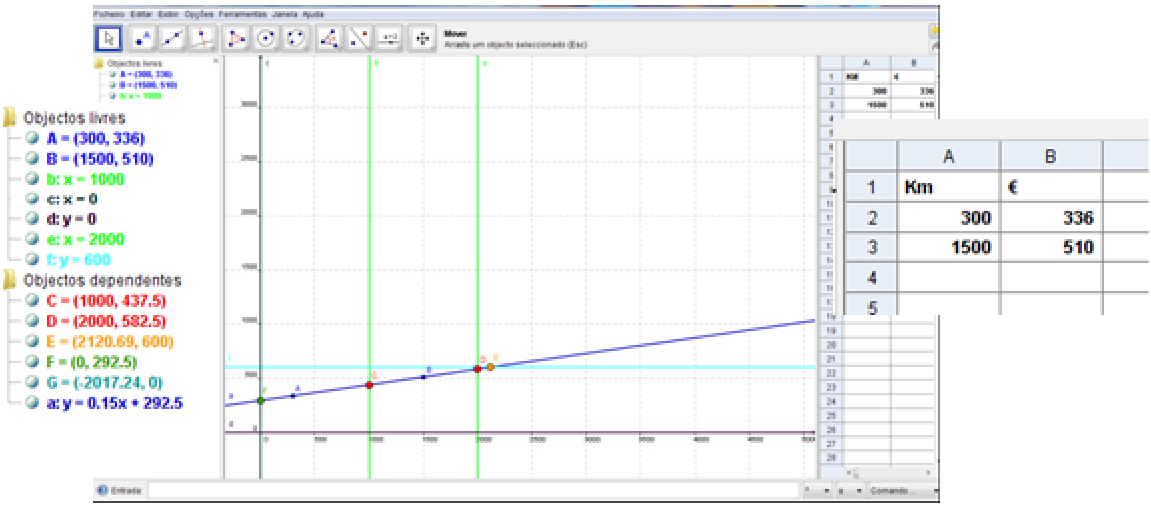

Figure 2: Model of the cost depending on the distance (screenshot of students' file)

To obtain an estimate of the monthly cost in the case of a travelled distance of 1000 km per month, students graphed the vertical line by entering the equation x=1000 and then determined the intersection point of the two straight lines. To calculate the cost for a distance of 2000 km, the procedure was similar to the previous one, leading to the coordinates of another point.

Students were also asked to determine the maximum distance not to exceed the monthly cost of 600 €. Soon they decided to plot the horizontal line of equation y=600 and, by looking for the intersection of the two lines, as happened in the previous cases, they answered to this question (figure 2).

Finally they interpreted the meaning of the y-intercept of the graph in the context of the problem. They conclude that, even if the car was stopped (0 km travelled), the owner had to pay expenses (292.5 €). The students were quite surprised at this result and gave it a contextual meaning as shown in the dialogue:

Diogo: It's the taxes, it's for the government!

Pedro: The meaning of this value? It means that I may not drive at all but I still have to pay at least 292.5 euros.

Task D: Speed in typing on a computer

The problem statement on the speed of typing a text on a computer included a graph with three linear functions plotted. To answer the questions, students would have to examine the graphs. They could just use paper and pencil or they could also use GeoGebra to solve the task.

> Ana, Beatriz and Carolina are learning to type a text on the computer. Their teacher tested their speeds and measured the time (in minutes) and the number of words written, obtaining the given graphs.
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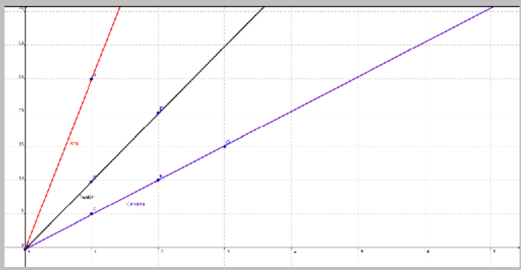
> 
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> 1. What is the fastest student?
> 2. How many words can each one write per minute?
> 3. How long does it take each of them to write a text with 520 words?

Pedro chose to plot the given graphs in GeoGebra, thus transposing the situation to the computer where he continued to develop his work. Then he explained:

Pedro: So far I plotted the lines shown in the problem in GeoGebra and then it gave me the equations of the lines. Now I will graph the line y=520 and intersect it with the previous lines to know how many words can they… (pause) how many minutes each of them takes to type 520 words.

This way, the student obtained the intersection points and gave the following answer:

Pedro: Ana takes 20.8 minutes, Beatriz takes 52 minutes and Carolina takes 104 minutes.

The teacher questioned the student on the use of GeoGebra in getting the equations for each case and checked if Pedro would be able to do it without the software. Pedro was one of the fastest students in using the software to obtain graphs from tabular data and reading out the algebraic outputs; he was also very alert to the possibility of entering an equation and getting the correspondent graph. So, he moved forth and back from one representation to the other and it was important to see if he was just relying on the tool or if the tool was offering him the chance to see the formal mathematical model embedded in multiple interconnected representations.

Teacher: Now tell me one thing, you chose to use GeoGebra but could you have done it without GeoGebra?

Pedro: If I had not done it with GeoGebra... (pause) then I would have to find the expression.

Teacher: How? For example, in the case of Ana, which is the expression?

Pedro: Well it is a straight line passing through the origin, so the model is y=kx. And as 1 corresponds to 25, we get y=25x.

CONCLUSIONS

Along the sequence of tasks, the formulation of mathematical models developed, in most cases, with the use of the computational tool. Mostly, the modelling process started with some numerical information and students used the Spreadsheet View to convert this tabular information into a graphical representation. The results show that students used the graphical representation of the mathematical models as the main source for developing their analysis of the models, which allowed them, among other things to *avoid* solving equations. The action of the students (plotting a graph) was followed by a prompt from the tool (the equation). This in turn generated new actions from the students: entering formulas to get vertical or horizontal lines. The tool then provided points of intersections and their coordinates; the students looked at the values that were displayed to find solutions for the equations and thus applying the model to find their answers. The way in which the computer was used illustrates an iterative process of co-action between the students and the tool. Moreover there is also action and reaction between the computer model, the mathematics world and the real context in providing meaning for the variables and for the algebraic expressions.

The case of Pedro indicates that, in many situations, the students were able to get the solutions algebraically. Therefore it seems that working with algebraic expressions and solving linear equations was not a strong obstacle to most of the students. The option for the geometrical manipulation of linear models is echoing previous research that highlights students' preference for non-algebraic approaches (Stacey & MacGregor, 2000). In our study, this kind of preference can also be explained on grounds that go beyond potential difficulties in algebraic manipulation. The affordances of the computational tool were assimilated by the students and reflected on their use of geometrical objects (lines, intersections, points, coordinates) to come across alternative approaches for exploring the model.

The geometrical representation became their object of reference in the modelling process and moreover it became a means to obtain the algebraic equation in the Algebra View. This seems to be a good example of the "algebracising" mode of the software used (Greefrath, 2011). In fact, at the beginning students were surprised with this utility provided by the software and quickly started to appropriate that in their exploration of the models. Similarly they realised that by entering an equation in the input bar (x=k, or y=k), the geometrical object immediately came up in the Graphics View. Consequently, this immediate translation from geometry to algebra and from algebra to geometry had an influence on students' modelling approaches. Regarding the application of models, the strategies used were essentially geometrical, taking advantage of the possibility of inserting new lines and relating the objects in the Algebra and Graphics Views. This is the main reason why many of the equations involved in the problems were solved from a geometrical point of view, by intersecting lines and obtaining the coordinates of the intersections.

The data reveal how students were guided by and simultaneously guided the computational tool to explore and understand linear models, showing the co-action between the student and the medium (Moreno-Armella & Hegedus, 2009) in the "technology world" of the modeling activity (Siller & Greefrath, 2010).

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