DEVELOPING A CRITERION FOR OPTIMAL IN MATHEMATICAL MODELLING

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Abstract: *Optimisation of real-world phenomena with the help of mathematics constitutes one direction of mathematical modelling. Thus, actions carried out while working on optimisation tasks will be located in the modelling cycle. Aspects of the concept of optimal are discussed on a meta-level. In order to identify processes of optimisation, some transcripts of students working on an optimisation task are analysed. Successively changing the perspective on the problem to be solved, starting from an individual and broadening to a collective (or economic) perspective, seems to be a successful approach which leads to an extraction of a suitable criterion for "optimal" in certain cases.*

Key words: *mathematical modelling, optimising, conception of optimal*

INTRODUCTION

The term "optimal" plays an essential role in everyday life, whether it is advertisement that praises a product's optimal features or practices in life that have to be improved or optimised. So, optimising is daily routine, although often on a subconscious level. Because one is confronted with the word, it is important to be able to reflect on its meaning. That contains asking in what sense something is optimal and – maybe more crucial – for whom it is optimal. Furthermore, it is necessary to know different aspects of the concept of optimal to be capable of optimising professionally or scientifically. Finding optimal solutions for real world problems with the help of mathematics is part of mathematical modelling. When Maaß (2010, p. 289) lists "objectives linked to the implementation of modelling", especially two items seem to be relevant in the context of a conception of optimal:

- "The students should be able to apply mathematics in their everyday life and their professional life.
- Mathematics is supposed to help students in understanding their world and in critically viewing mathematical information in the sense of active citizenship."

The Duden (2012), a dictionary of German language, describes the word "optimal" as *best possible under given restrictions, with regard to a goal to be achieved* [1]. These are also main aspects of the notion of optimal in mathematics. The challenge lies in capturing restrictions as well as the goal and in translating it into mathematics. Modelling tasks which encourage an examination of the word "optimal" should contain questions where neither the restriction nor the goal is obvious a priori. Restrictions, on the one hand, can often be determined by applying assumptions and collecting additional data. On the other hand, the goal to a high degree depends on subjective and intuitive assessment of the situation and on the perspective from which the situation is considered. Therefore, the goal has to be negotiated with participants. For some goals this might be more complicated than for others. For example, cost minimisation can be considered as a clear goal where a consensus is found soon (except from sources of costs). In contrast, striving for maximum fairness can be expressed in many different ways which requires reasonable compromises.

This paper argues that certain modelling tasks can foster the development of a conception of optimal. Furthermore, an analysis of students working on optimisation tasks suggests that there are ways of elaborating a criterion for "optimal" that seem to be more successful than others.

THEORETICAL BACKGROUND

Schreiber (1979) describes *optimality* as a fundamental idea in the Bruner sense. In this context optimality is seen as property of forms, variables, numbers etc. satisfying a given condition best possibly (Schreiber, 1979, p. 167, my translation). While this highlights the relevance of an attributing (static) meaning of optimisation for mathematics education, Schupp (1992) identifies *optimising* as fundamental and focuses on the procedural (dynamic) meaning. In formulating levels of optimising he emphasises the last level called "meta-optimisation", in which a reflection on optimising with students takes place. This includes besides others the reflection on solving processes, strategies as well as on intra- and extra-mathematical meaning of optimising (Schupp, 1992, p. 114). In the context of the reflection on the best solution method it is hinted that a decision concerning this matter depends on the situation, the criterion, the question what is to be understood as "better" (or best), and on the individual preferences (Schupp, 1992, p. 159). In this sense this perspective on optimisation includes uncertainty and a need for critical considerations of decisions and results which is characteristic for mathematical modelling in general.

Understanding optimising as one direction of mathematical modelling, the actions of optimising can be located in the modelling cycle (see figure 1). Relevant conditions and restrictions have to be identified and structured. These conditions and restrictions can be expressed in a *real model*. But a real model requires even more: Finding a suitable perspective on the problem or weighing up different perspectives can be considered as *simplifying and structuring* and takes place in the *rest of the world*. Is there a certain perspective extracted and an objective criterion for optimal is expressed, a *real model* was created. Objective criteria are considered as those, where anybody else would decide on the fulfilment of the criteria in the same way. The translation of the criterion and the conditions into mathematics leads to a *mathematical model*. Working mathematically provides *mathematical results* which have to be interpreted to get *real results*. Validating concerns the correspondence between real results and the situation (Borromeo Ferri, 2006). This may lead to acceptance or refusal of the solution.

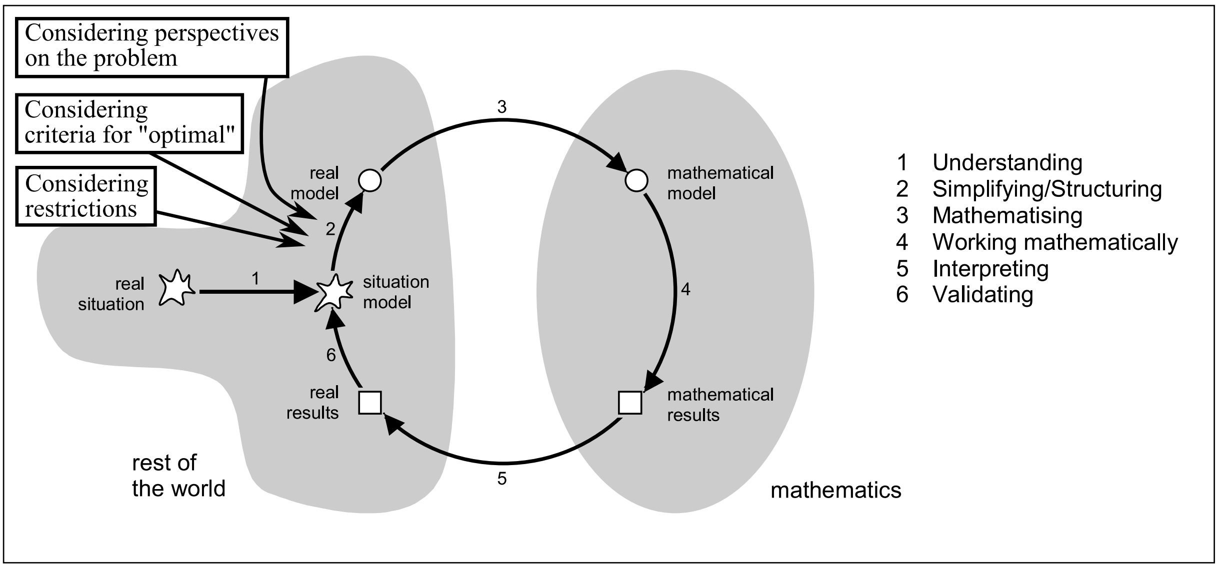

Figure 1: Main aspects of optimising as part of the Modelling Cycle by Blum and Leiß (2006, p. 1626).

In a single optimisation task there are several ways to deal with different perspectives. While above it is described that one perspective has to be extracted, it is also possible to keep all perspectives in order to mathematise them separately. This allows a comparison of the perspectives concerning the results they lead to. Some perspectives may lead to the same result, so that an extraction of one perspective is not necessary.

Especially in steps 1 to 3 and in steps 5 to 6 of the modelling cycle meta-knowledge on optimisation is needed, in the first steps in order to realise the need of finding different perspectives and in the last steps in order to be able to critically question an "optimal" solution in a mathematical sense. Furthermore, results from validating may provoke a shift in perspective which leads to a revised real model.

METHOD

The analysis is based on data which was gained by Busse (2009). In the context of his PhD-thesis four pairs of 16-17 year old German students were videotaped while working on modelling-tasks. Afterwards each student watched the video record that was showing him or her. In doing so the students were asked to verbalise thoughts concerning the real-world context which emerged while working on the tasks. Both the student and the researcher could interrupt the video record to initiate this verbalisation. This so called stimulated recall was also recorded. The transcripts concerning one of the modelling-tasks (see Fig. 2) are analysed here. Although the task does not mention optimisation, it constitutes an optimisation task as the question on the "best position" of the common house is implied.

In a little wood a home for aged people has been built. In the figure the seven residential buildings are marked by black dots. There are paths in the wood so that the aged people do not have to walk through the undergrowth. The paths are marked by bold lines. On the path between the two crossings (marked by a dotted line) a common house is planned to be placed. This common house is meant to serve for afternoon coffee and evening events. The question is where exactly on this path the common house is to be built.

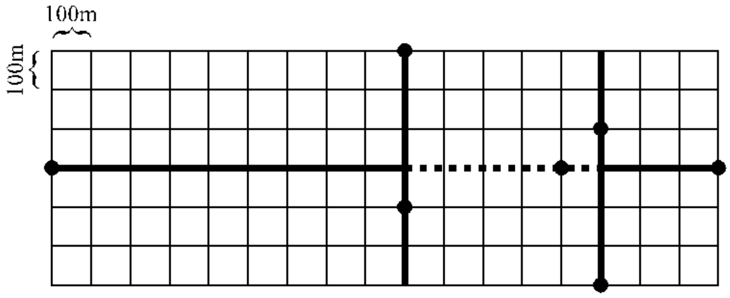

Figure 2: Modelling-task "Home for Aged People" (Busse 2011, p. 40).

The analysis includes three steps:

- Text segments concerning a *certain perspective on the task*, a *criterion for optimal*, a *restriction* or a *solution* are extracted. These aspects can be on an explicit or implicit level.
- The working process is traced by considering the aspects mentioned in the first item.
- An abstraction from the working process is made.

PRELIMINARY RESULTS

As a first approach towards a solution, three pairs of students state that the common house *can be placed anywhere*. The remaining pair expresses negative knowledge by considering that the common house *should not be placed at the right crossing*. Whereas in the first case no criteria for a "good" place for the common house seem to be available, there are hints for such criteria in the second case, although implicit and vague. For example, one criterion could be a low maximum range from the residential buildings to the common house. These different utterances do not permit judging on the student's work but they rather constitute an approach to the task which concludes that there is a lack of information.

In further examination of the problem the four pairs proceed in a very different way. One pair is first irritated by the lack of information and needs hints from the researcher. Then these students intuitively mention a criterion and a solution which they accept without or with an unconscious and intuitive validation in the sense described by Borromeo Ferri (2006). It takes them less than five minutes to finish the task so that the full potential of the task is not taped. The chosen criterion and the matching solution seem coincidental.

The question is what is necessary to accept the working process as suitable, in which way criteria can develop and how these criteria can influence a solution. This is described in the following by having a closer look on another pair's working process. Its members Heinrich and Ingo reflect on different perspectives and matching criteria intensively.

After reading the instruction Heinrich spontaneously states a criterion [2]:

51 H: Oh. So that everybody has the lowest possible distance.

This focuses on the distance between the residential buildings and the common house and expresses a possible need of every single aged person. In this way the problem is regarded from an *individual perspective*. The fact that a shorter distance for one person could mean a longer distance for another person constitutes a conflict caused by this criterion which is not seen consciously at that time. Heinrich repeating this statement in variation (line 100) a restriction comes to the fore soon:

139 H: But I'm assuming that as little as possible uh- so, that all have [...] a distance that is fair.

So *fairness*, which in the stimulated recall turned out to be seen as equality, dominates the working process from now on. In the stimulated recall Heinrich verbalises his thoughts:

154 H: [...] that if possible all people will have the same distance- or better, so that, that it is not unfair [...] that not most of the people have a short distance and one person has a very long distance.

This highlights the relevance of the residential house at the left end which will be furthest away from the common house. In order to reduce the longest distance between the residential buildings and the common house Heinrich votes for the common house to be at the left crossing (line 741). Referring to this position for the common house he concludes:

769 H: Actually this is fairest. Because then the longest distance is nine hundred [pointing at the house at the left end].

This perspective can be called a *group-related perspective* because certain groups – here groups that are disadvantaged – are considered. It constitutes a shift from the individual perspective towards a more abstract one.

Whereas Heinrich is the more active in the promotion of this solution Ingo brings into account a new perspective. This perspective could be described as a *collective or economical* one. He states:

915 I: We can also do it so- uh that you uh- test it for all and then you take the sum of the distances to all houses [...] and where the sum is lowest.

927 I: [...] Look. You sum-up all distances from every house. [...] Distances to the house in the middle. You sum it up. Then you have a sum. And you make this for every possible position of the [common] house. That means six all in all.

944 I: [...] Then you compare the sums and where the total distance is lowest-

948 H There it is best. In fact that’s true.
951 I: But the problem is that this is not fair.
981 H: No, that’s not fair.

This point of view can be described as a *collective or economical perspective* because individual needs remain unconsidered. Instead, it is argued in a more unemotional and economical way. A time-consuming calculation that is carried out by the students results in the best position for the common house being at the same position as the house on the dotted line. A long discussion on whether the one or the other solution is the most suitable leads to the acceptance of the first one. In the stimulated recall it is argued (line 955) as indicated in the transcript above that the second solution might be good in other contexts, but in the context of aged people where long distances are covered laboriously it is refused.

Reference back to aspects of the concept of optimal

In the case of Heinrich and Ingo the solving process includes main aspects of the concept of optimal. On the one hand, that means an intensive discussion on the aim of an optimal solution by considering different perspectives, here called individual and collective perspective. On the other hand, the conditions and restrictions were taken into account thoroughly.

Abstraction of the case: From individual to collective perspective – choosing the “right” one

A natural behaviour in determining which solution works best is to wonder which solution would work best for oneself, assuming oneself being one of the persons affected by the decision to be made. Concerning the *Home for Aged People* task one consequence of this perspective could be:

> Everybody wants to have the shortest (or longest – if noise is expected) possible distance between his residential building and the common house.

A conflict occurs when *better* for one person means *worse* for another person. This can be the case in many situations when several people are involved. Therefore, a new perspective is necessary. From the perspective of involved people that are disadvantaged, a group-related criterion is generated that diminishes disadvantages. This criterion can be expressed as:

> The longest distance should be lowest possible.

Changing to a more collective and economic perspective may lead to the following statement:

> The sum of distances of all aged people should be lowest possible.

Successively the perspective changed from individual criteria via criteria of certain groups to a rather collective and economic criterion where individual needs remain unconsidered.

Solely applying the collective and economic criterion brings a wide difference in distances. One person would have to walk much longer than another. Under the condition of fairness – which is seen as equality – this criterion seems unsustainable for the students so that the group-related perspective is considered as the “right” one.

It should be mentioned, that not one perspective is of higher quality than another per se – it is worth considering the needs of every single person (individual perspective) – but considering all perspectives as a whole constitutes a great level of understanding of the situation.

CONCLUSION

Mathematical modelling is challenging for students. Managing the lack of information is one obstacle. Optimising provides further difficulties, for example, the choice for a suitable perspective is not clear a priori. The transformation of the perspective from which the pair described looks at the problem suggests that this proceeding fosters successful optimising and therefore can be used as strategy in the rest-of-the-world-part of the modelling cycle. The transformation of the perspective can be characterised as changing successively from individual to collective needs with respect to existing restrictions.

However, there might be other determining factors and strategies that provide good results. In further studies it should be explored, if the strategy presented here turns out to be sustainable in other optimising tasks as well and if other determining factors and strategies can be identified. The impact of student’s meta-knowledge about optimising is a question that has to be explored likewise.

Furthermore, sources for the shift of perspectives should be explored. That contains the question on what causes a change of perspective.

NOTES

1. This has been translated from German into English by the author.

2. These are parts of the transcripts produced by Busse (2009). They partly have been translated from German into English by the author. For the reason of better readability, pauses and accentuations are not displayed. Names have been changed.

REFERENCES

Blum, W., & Leiß, D. (2006). "Filling up" - The problem of independence-preserving teacher interventions in lessons with demanding modelling tasks. In M. Bosch (Ed.), Proceedings of the 4th Congress of the European Society for Research in Mathematics Education (pp. 1623–1633). Sant Feliu de Guíxols: Universitat Ramon Llull.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. ZDM, 38(2), 86–95.

Busse, A. (2009). Umgang Jugendlicher mit dem Sachkontext realitätsbezogener Mathematikaufgaben: Ergebnisse einer empirischen Studie. Hildesheim, Berlin: Franzbecker.

Busse, A. (2011). Upper Secondary Students' Handling of Real-World Contexts. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), International Perspectives on the Teaching and Learning of Mathematical Modelling: Vol. 1. Trends in Teaching and Learning of Mathematical Modelling. ICTMA14 (pp. 37–46). Berlin: Springer.

Duden. (2012). Duden online. Retrieved from http://www.duden.de/node/650218/revisions/650223/view

Maaß, K. (2010). Classification Scheme for Modelling Tasks. Journal für Mathematik-Didaktik, 31(2), 285–311.

Schreiber, A. (1979). Universelle Ideen im mathematischen Denkne - ein Forschungsgegenstand der Fachdidaktik. Mathematica didactica, 2, 165–171.

Schupp, H. (1992). Optimieren: Extremwertbestimmung im Mathematikunterricht. Mannheim: BI-Wiss.-Verl.