MEANINGS OF THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT ONE POINT: BACKGROUND AND ADVANCES

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In this paper we present a description of the previous works carried out by authors from the general issue of designing and implementing a didactical planning for Spanish students from Non-Compulsory secondary education, 16-17 years old. The current research has as aim to describe the meanings that students associate to specific terms from the language, such as, "to approach," "to tend,""to reach,""to exceed," and "to converge." Prior to the study, we reviewed the mathematical use of these terms and contrast this employment with their colloquial use. From the semi-structured interviews used to gather information, we provide the analysis of the written data. It is important to highlight that students have contributed with a variety of meanings, in addition to those from the previous review.

Keywords: Conceptual analysis; Finite limit of a function at one point; Meaning of a mathematical concept; Misconceptions; Difficulties and errors.

INTRODUCTION

Since academic year 2009/2010 we have been interested in investigating some problems related to the teaching and learning of the concept of limit of a function at one point. This concept is important because is necessary for the learning of the derivative and integral concepts and overcomes in complexity the concept of limit for sequences. Furthermore, it is one of the key concepts that mark off the transition towards the advanced mathematical thinking. By exploring several textbooks we observed a high amount of routine tasks of calculation of the limit on the base of an intuitive definition based sustained on the idea of approximation. So we carried out an exploratory study about the intuitive meanings that students have about the concept of finite limit of a function at one point when they are questioned about several tasks using different representations such as verbal, graphic and symbolic representations (Fernández-Plaza, 2011). Some of the results have been presented both in several national and international conferences (Fernández-Plaza, Ruiz-Hidalgo & Rico, 2011, 2012a, 2012b, 2012c).

Recently, we have gathered information by means of interviews in order to contrast our interpretation of the written records from students and to deepen in the personal conceptions that students associate to the following terms from calculus: "limit," "to approach," "to tend," "to converge," "to reach" and "to exceed," whose colloquial meaning influent in understanding and were reported in several studies (Monaghan, 1991; Cornu, 1991). In Fernández-Plaza (2011), we have explored the effective use of these terms and others synonyms, not the specific meaning implemented by students. By effective use of a term, we mean that students use in fact that term, not a synonym. For example, for the specific term "to approach", a student may use "to get close" or "to approximate," among others, which are not effective uses but related to "to approach."

Below, we describe the main achieved results.

MAIN ACHIEVED RESULTS

We summarize the most important results we have found out until the present moment.

Firstly, we observed a persistence of misconceptions related to the limit as an non exceedable and unreachable value. This result is consistent with those from Cornu (1991) and Monaghan (1991), we go beyond, in the sense that, some students suggested a link between exceedability and reachability. We consider that this kind of misconceptions could arise from an over generalization of the particular case of monotone convergence.

Secondly, we discriminated between *process conceptions*, *object conceptions* and *dual conceptions* of concept of limit. As *process conceptions*, we understand those ones considering limit is closely related to one procedure to find it; as *object conception,* student is able to identify properties of the limit without depending on the process involved; intermediate conceptions between these two are called *dual conceptions.* Thus when students were requested to discuss about the statement "The limit describes how a function f(x) moves when x moves to certain point," the most of arguments could be classified as one of these three options depending on whether students interpreted the limit as "how" (process conceptions) or "where" (object and dual conceptions) a function moves.

Thirdly, we found out conflicts with the arbitrary precision of approximation to the limit. Thereby, the expression "limit can be approximated as much as you wish" provoked that some students affirmed that the practical process is finite, so precision too. We appreciate that they do a crucial distinction between the potential infinity character of the process and its implementation in practice

Finally, we pointed out the conflicts with the exact or indefinite character of the limit value.

Some subjects considered a limit as an exact number versus others that considered that the limit is an "approximated" number. We suggest according to Sierpinska (1987) that the latter subjects do not know what the limit is, but only approximations.

The progressive improvement in the interpretation of these results gave rise to talk about *structural aspects,* defined as those characteristics, properties, notions and terms, documented in literature. These structural aspects were used to characterize and establish connections between different conceptions about the concept of limit (Fernández-Plaza, Ruiz-Hidalgo & Rico, 2012a). Parallelly, we tried to characterize the terminology used by students to explain the previous properties about the concept of limit. Among other reasons we selected the following terms "to approach," "to tend," "to converge," "to reach" and "to exceed" because they are used in the technical language and describe different aspects of the concept of limit. Moreover and the influence of their colloquial meanings and everyday uses on students' understanding has been reported in the literature. This problem leads us to three questions:

- Which are the different meanings and uses that these terms have in Spanish language?
- What is the terminology that students effectively use to explain their answers?
- What is the effective meaning that students associate to these specific terms?

The treatment of the two first questions can be consulted in Fernández-Plaza (2011) and Fernández-Plaza, Ruiz-Hidalgo and Rico (2011, 2012b). In the following section we are going to focus on giving answer to the third question.

DESCRIPTION OF THE CURRENT STUDY

We propose to describe how students define explicitly some specific terms from calculus in contrast with a review of meanings of these terms. These terms are "to approach," "to tend," "to converge," "to reach" and "to exceed."

Theoretical Framework and prior research

We position this study in the research agenda of *Advanced Mathematical Thinking*, from the international group on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006, pp. 147-172). We assume the difficulty of delimitate the transition from elementary to advanced mathematical thinking.

The educational stage analyzed assumes a period of transition in which students use elementary techniques to tackle mathematics contents whose development historical, epistemological, and didactic have an advanced status.

We assume the notion of the meaning of a mathematical concept developed by Rico (2012), based on reference, sign and sense. We analyze the systems of representation, formal aspects or references of the concept, and the phenomena that provides its meaning.

Three components are on the basis of the meaning of a mathematical concept:

- *Systems of representation*, defined by a set of signs, graphics and rules, to make the concept present and to establish relationships with other concepts.
- The *conceptual structure* that comprises concepts and properties, the derived arguments and propositions and their verity criteria.
- *Phenomenology* that includes those phenomena (contexts, situations or problems) which are in the origin of the concept and provide it sense. (Rico, 2012, pp. 52-53)

The mathematical language related to the concept of limit of a function at one point includes the terms "to approach", "to tend", "to reach" and "to exceed". We chose these terms, because among other reasons that may be consulted in Fernández-Plaza (2011, pp.14-21), each of them refers in part to properties and modes of usage associated with the concept of limit, that is to say, the phenomena involved according to the notion of meaning above mentioned.

Conceptual analysis of specific terms

By conceptual analysis we understand the procedure leaded to establish the mathematical use of the terms and to contrast this employment with their colloquial use or use in other disciplines.

We describe these terms below and also include the colloquial meaning of the term "limit."

The sentence "to tend toward a value" means "to approach gradually but never reach the value" (RAE, 2001) and expresses a very specific form of approach. Blázquez, Gatica and Ortega (2009) argue that a sequence of numbers approaches a number if the error decreases gradually, but they also argue that a sequence "tends toward a limit" if any approach to the limit can be measured by the terms in the sequence. We establish a distinction between these two terms.

A study by Monaghan (1991) concludes that many students do not distinguish between "tend toward" and "approach" in a mathematical context.

The correct use of the term "to tend toward" should be determined using the variable x and not f(x), since the expression "f(x) tends toward L, when x tends toward a" may cause cognitive conflicts, as Tall and Vinner (1981) note.

"To reach" means intuitively "to arrive at" or "to come to touch" (RAE, 2001; Oxford, 2011). We interpret "reach" as meaning that a function reaches the limit if the limit value is the image of the point at which the limit is studied (continuity); by extension, the limit can be the value of any other point in the domain.

We see that "to exceed" means colloquially "to be above an upper level" (RAE, 2001), excluding the meaning "to be below a lower level". We will say that the limit of a function may be exceeded if we can construct two successive monotones of images that converge at the limit, one ascending and the other descending, for appropriate sequences of values of x that converge at the point at which the limit is studied. The reachability or exceedability of the finite limit of a function can be easily interpreted as global or local concepts, but there is no logical implication of the two concepts.

The term "to converge" means colloquially "to come together from different directions". In mathematics, this term is equivalent to "to tend" and normally is applied to the limit of sequences and series; however, it is not frequent to use this term with the limit of a function at one point. We expected that students could invent a definition for this term in this new mathematical context.

Furthermore, the term "limit" has colloquial meanings that interfere with students' conceptions of this term, such as ideas of ending, boundary, and what cannot be exceeded (RAE, Oxford, op. cit.). The term's scientific-technical use is related in some disciplines to a subject matter or extreme state in which the behaviour of specific systems changes abruptly (RAC, 1990).

Prior Research

Monaghan (1991) studies the influence of language on the ideas that students have about previous terms, as these terms are employed with different graphs of functions and the examples that school students verbally explained. We underline as a limitation the approach adopted in this case, in which the key terms that the students were asked to use were defined a priori, instead of enabling students to use their own words freely and spontaneously and to infer the appropriate nuances a posteriori.

In CERME proceedings have been published some works related to the learning of the concept of limit of a function. The most related to this study is Juter (2007) who investigated, among other aspects, how students interpreted the reachability of the limit in a problem solving context and theoretical discussion.

Method: Instrument

A semi-structured interview was conducted in an ordinary classroom. The protocol of implementation was the prior request to the students to write their answers in the facilitated sheet, to the discussion of the answers that was audio recorded. The subjects were organised into nine groups with 3-5 components, in order to facilitate the interaction between the subjects and the researcher.

We focus on the following common question:

Describe in each gap how you understand the following terms: "to approach", "to tend", "to reach", "to exceed", "to converge" in the context of finite limit of a function at one point.

In order to help students to better express their conceptions during discussion, we showed them some graphics of functions so that some other characteristics of meaning of these terms could emerge, especially with the terms "to reach" and "to exceed".

Method: Subjects

33 subjects out of a total of 36 subjects from the previous study (Fernández-Plaza, 2011) were selected. They were chosen deliberately, according to their previous answers and based on their availability. The subjects were studying the first year of non-compulsory secondary school study, 16-17 years of age, who were taking the subject of Mathematics.

Preliminary results and discussion

We are going to show some preliminary results from the analysis of the written records. The following table 1 shows the different kinds of meanings and frequencies about the specific terms that subjects provided on their answer sheets.

Specific terms	Meanings	Frequencies
To approach	A1. To get close as possible	15
	A1.1. Not to reach the limit	11
	A1.2. Not to reach and not to exceed the limit	1
	A1.3. To reach but not to exceed the limit	2
	A2. To establish the closest value to the limit	2
	A3. Other	1
	A4. No answer	1
To tend	B1. To approach	9
	B1.1. Not to reach the limit	5
	B1.2. To reach the limit	2
	B2. Technical usage	8
	B3. Subjective	4
	B4. Other	4
	B5. No answer	0
To reach	C1. To arrive at or to touch the limit	24
	C1.1. Not to exceed the limit	3
	C2. To know the exact value of the limit	2
	C3. To know the value of f(x) for a given x	1
	C4. Other	1
	C5. No answer	2
To exceed	D1. To surpass the limit of the function f(x)	19
	D2. To surpass the x-value.	6
	D3. To reach the limit and continue (To pass through the limit or x-value)	3
	D4. Other	0
	D5. No answer	3

	E1. The function is above the limit all the time	3
	E2. The function is below the limit all the time	2
	E3. To tend	1
	E4. To reach	1
To converge	E5. The right and left-hand limits are the same.	5
	E6. The function takes the same value than the limit	1
	E7. Two functions or straight lines intersect at one point	6
	E8. Other	4
	E9. No answer	9

Table 1: Classification of meanings and frequencies about the selected specific terms

Below, we exemplify answers from some categories in order to clarify their denomination. The other categories are denoted by a "representative" definition, so we do not consider necessary to exemplify all of them:

- **Category B2: Technical usage.** A sample answer is "*This term is used to indicate the value that x takes in a limit*". It does not mean anything about the specific meaning of the action "to tend," that is, it is only a technical word; an agreement. Another answer is "*to tend to a number is to use the closest number to it, for example, if $x \to 1$ by the left hand, we use 0.9. By the right hand, we use 1.1.*" The term is used to describe a personal rule to calculate a limit.
- **Category B3: Subjective.** Two sample answers are: "*To approach to that number without being aware of it (without pretending it)*" and "*To approach it as much as we* want" indicate a subjective aspect of the definition of the term "to tend."
- **General category: Other.** We included answers which are not coherent or whose category is not clearly defined.

From Table 1 we discuss the global results:

Most of subjects (15 out of 33) consider "to approach" as *to get close as much as possible*. Although it is relevant that 11 out of 33 subjects go beyond considering explicitly that the function cannot reach the limit. Only 2 out of 33 admitted that the function can in fact could reach the limit but never exceed it. In general, "to approach" is considered as an intuitive and incomplete process. It is relevant that 9 out of 33 consider "to tend" equivalent to "to approach" and 5 out of 33 add the not reachable character.

However, the term "to tend" has some particular characteristics, as different from "to approach", such as, a subjective view of its definition (4 out 33) or a technical usage (8 out 33) , that is, it is an agreement in mathematics.

In regard to “to reach", most of subjects (24 out of 33) consider it as simple as *to arrive at or to get to touch the limit*. Only 3 out 33 considered that the limit must not be exceeded. In the other hand, only two subjects considered that the limit is reachable if we can calculate the exact value, while only one subject stated that “to reach” is to know the value of f(x) for a given x, so there could be a possible identification between the limit and the image.

“To exceed” is basically *to surpass the limit or the x-value given* (19 and 6 out of 33), although some subjects (3 out of 33) gave more complete answers, in a sense that a limit or x-value given are exceeded *if the function reaches them and continues, both above and below them*.

At the beginning, students recognised not to know the term “to converge” in the context of finite limit of a function at one point. In fact, 9 out of 33 did not answer this question, so the researcher had to encourage them to write whatever they imagine about any other situation and to invent a definition. Thus, only one subject defined this term as “to tend”, and most of the subjects (6 out 33) considered the meaning *two functions intersect at one point*, and 5 out 33 define “to converge” as *the lateral limits are the same*, definition that could be considered suitable in this context. On the other hand, several subjects described situations where the *function is all the time above or below the limit*, that is to say, an asymptotic behaviour of the function, for example, $f(x) = 1/x$ converges to 0 when x tends to ⁀∞.

Preliminary conclusions

According to the discussion of the results from the table 1 and the aim proposed at the beginning: To describe how students define explicitly some specific terms from calculus in contrast with a previous conceptual analysis of these terms, we draw the following conclusions about their achievement:

Students interpret in many different ways the meaning of the selected terms, most of them extracted from everyday situations, so we agree with Monaghan (1991) and Cornu (1991) that conflicts between colloquial and formal language are still occurring.

The review of the uses of specific terms has predicted partially the meanings that students were going to provided, above all the colloquial meanings. The technical use of the term “to tend” had been conjectured by Fernández-Plaza (2011, p.36). The observed difficulty of students to distinguish between “to approach” and “to tend” is consistent with Monaghan (1991) and Blázquez, Gatica and Ortega (2009). All the new meanings of these terms should contribute to enrich this review in order to increase its explicative power.

At the beginning, the term “to converge” had been considered unknown by students in the context of finite limit of a function at one point, but they had been able to invent a possible definition for the new context.

It is relevant that exceedability and reachability of the limit are especially connected to the students' conceptions of the terms "to approach" and "to tend" according to Fernández-Plaza (2011, p. 40).

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