PERCEIVING CALCULUS IDEAS IN A DYNAMIC AND MULTI-SEMIOTIC ENVIRONMENT- THE CASE OF THE ANTIDERIVATIVE[1]

Osama Swidan

University of Haifa, Haifa, and

Mandel Leadership Institute, Jerusalem, Israel.

The present case study was designed to analyze the objectification processes of making sense of the antiderivative concept when it is being studied graphically in dynamic and multi-semiotic technological environment. This study is guided by sociocultural theory, which considers artifacts to be fundamental to cognition and views learning as the process of becoming aware of the knowledge that exists within a culture. The case study focuses on two seventeen- year- old students. In the course of the discourse micro-analysis I identified three essential foci in the objectification processes 1) objectifying the relationship between a function and its derivative 2) objectifying the relationship between a function and its anti-derivative 3) objectifying the vertical transformation of the anti-derivative graph.

INTRODUCTION

The integral concept is considered to be central to learning calculus and I cannot imagine any curriculum in Calculus not containing the integral concept. The integral consists of two essential concepts - antiderivative and definite integral. The fundamental theorem of calculus connects these two concepts. Therefore, significant learning of the integral concept must encompass the learning of the conceptual aspects of both these concepts (Thompson, Byerley, & Hatfield, in press). (Thompson & Silverman, 2008) suggested introducing the accumulation function as a tool connecting these two concepts. Thompson et al. (in press) have proposed a didactical sequence that serves to emphasize the conceptual aspects of the integral concepts when these are taught to college students as the accumulation function, by means of technological tools. To improve our understanding of the processes involved in learning the integral as accumulation graphically among high school students, follow Thompson & Silverman (2008) and Thompson et al. (in press) and propose a learning unit composed of five sessions. This learning unit is based on graphic and numeric signs and dynamic interfaces, which consider the graph of the accumulation function as central in conceptualizing the integral concept. Yerushalmy & Swidan (2012) have analyzed and identified processes of making sense of the definite integral as accumulation in a dynamic and multi-semiotic environment among high school students.

[1] This case study is part of a PhD study under supervision of Prof. Michal Yerushalmy

Few research studies have examined the learning of the antiderivative concept in general, and among high school students in particular. For example, (Haciomeroglu, Aspinwall, & Presmeg, 2010) have examined the mental processes and images used by three university students to create meaning for graphs of the derivative. (Berry & Nyman, 2003) have studied college students' understanding of the relationship between the derivative and the antiderivative function while drawing graphs of functions from graphs of the derivative by using a sensory calculator (calculator-based ranger[2]). The scarcity of studies dealing with the process of learning of the antiderivative concept by high school students and the usefulness of graphical tools such as graphic interfaces in teaching the conceptual aspects of calculus concepts is what motivated us to designs the current study. This study aims to analyze the processes involved in high-school students' learning of the antiderivative concept graphically as accumulation function, involving graphical and dynamical tools.

THEORETICAL FRAMEWORK

According to sociocultural theory artifacts of any kind are central and play a fundamental role in cognition. It has been claimed that, within the social use of an artifact to accomplish a task, shared signs, which relate to the artifact, are produced and may be related to the content intended to be learned (Bartolini Bussi & Mariotti, 2008). The relationship between an artifact and knowledge is expressed by culturally determined signs. The relationship between an artifact and accomplishing a task is expressed by signs such as gestures, speech and drawing.

Signs in general, and mathematical signs in particular, play two roles. (L. Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005) define these roles as "social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (p.117). (Berger, 2004), who studied the functional use of mathematical signs, suggests a twofold interpretation of the meaning of signs and objects: a personal meaning, "to refer to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not)," and a cultural meaning, "to the extent that its usage is congruent with its usage by the mathematical community" (p. 83). In the context of using artifacts, Bartolini, Bussi and Mariotti (2008) describe the relationship between the personal and the mathematical meaning as a double semiotic relationship: "On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use." (p. 754). Adopting these terms, I define the double semiotic relationship as the semiotic

---

[2] The Calculator based ranger is a motion sensor device that can collect and represent real-world motion data, such as distance or velocity.

potential of an artifact, and assume that the potential is defined with respect to a particular design and pedagogical goals.

Learning in this setting means participating in an active process that leads to making sense of the elements, and bringing about an encounter between personal and mathematical meanings. In other words, to learn something, the learner must attend the existence of the knowledge within the culture and become aware of its existence (Radford, 2003; Radford, Bardini, & Sabena, 2007). The attention and awareness processes of an existing mathematical object require engagement in a mathematical activity to grant meaning to the object. Radford (2003) called this process an objectification process. Objectification requires making use, in a creative way, of different semiotic tools such as words, symbols, and gestures available in the universe of the discourse (Radford, 2003). Semiotic tools play a central role in the objectification process. The design of the present study, the artifact and the tasks, allow us to study the semiotic potential of the artifact and to analyze actions that make the double semiotic relationship related to the antiderivative concept observable.

THE MATHEMATICS, PEDAGOGY AND SEMIOTICS OF THE ARTIFACT

The artifact used in this study was 'The Calculus Integral Sketcher (CIS)' (Shternberg, Yerushalmy, & Zilber, 2004) (Fig. 1). As a multi-semiotic system, the CIS contains different types of signs that I grouped into two categories:

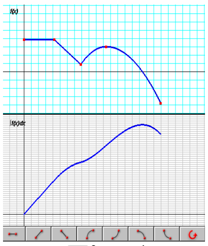


Fig. 1

1- ***Cartesians Graphing system:*** Two Cartesian coordinate systems, one above the other, coordinated horizontally. The curve in the upper Cartesian system signifies a function f. The curve in the bottom system signifies the values of $\int_a^x f(u)du = g(x) - g(a)$ where the derivative of g(x) is the function f(x).

2- ***Iconic Graphing tools***: students choose the graph in the upper Cartesian system by pressing an icon at the bottom of the CIS. Schwartz and Yerushalmy (1995) describe this set of icons as a mediating language for modeling consisting of segments from a wide range of single variable functions.

Sketching a function graph [f – in fore coming] in the CIS is carried out by choosing an icon from the icons applet, placing it into the upper Cartesian system, and manipulating it by dragging the segments or their end points. The sketch in Fig. 1 shows a curve drawn by using multiple icons (linear and nonlinear), and its antiderivative graph [g – in the coming] is drawn by the CIS in the lower system. The tool allows dragging ‘f’ freely and the ‘g’ changes accordingly. Dragging the ‘g’ is permitted only vertically and in this case, ‘f’ is stable in the upper Cartesian system.

THE DESIGN OF THE STUDY

This case study is part of a longitudinal study that aims to analyze the learning processes of the integral concept. The study explored about one hour of learning by Mohamed and Ahmed, two seventeen-year-old students in the mathematics class taught by the author. The episode took place at the computer lab at the students’ school. At the time of the episode, the students had already acquired the concepts of function and derivative, but not that of the integral[3]. The students were familiar with using the derivative symbolically. The two students shared a single computer, and the researcher introduced them briefly to the interface.

To explore the processes of making sense of the antiderivative graphically, I asked the students to interpret the mathematical relationships between the graphs that appeared on the computer screen. To cover the majority of the cases of one variable functions, I asked the students to create graphs using either a single icon or multiple icons. The students were given the following instructions:

> To complete the task, you will use the Integral tools of CIS. You are invited to create graphs of signal icon and multi icons. Your task is to come up with a conjecture and explanation about the mathematical relations between the upper and lower graphs. You can work as long as you want, until you feel that you can sketch a graph without any tool that will appear in the lower window for a given function graph.

The students were video-recorded and their computer screens were captured. The video recording was achieved by software which captured the footage in two different windows: the computer screen and the students' body. The researcher was present as an observer and available to provide technical and miscellaneous clarifications.

DATA ANALYSIS

I used *attention* and *awareness*, Radford's (2003) categories of objectification of knowledge, to analyse the evolving processes of personal and mathematical meanings. I identified *attention* as a declaration about the existence of a mathematical relationship between objects in the semiotic system. In the present study, declarations about the existence of mathematical relationships tended to be based on visual considerations. Justifications and interpretations based on mathematical considerations of the mathematical relationships students had noticed were defined as *awareness*.

---

[3] They had encountered the integral symbol and its computational uses in their physics and electronics lessons

I present here the second round of analysis. The first round consisted of reiterative watching of the video, concentrating on students' actions with the tools, their repeated gestures, argumentation, and interpretations. The second round involved searching for and classifying the transcripts into three main categories which were taken from the data: 1) objectifying the relationship between a function and its derivative 2) objectifying the vertical transformation of the antiderivative graph 3) objectifying the relationship between a function and its antiderivative. Therefore, I collated chronologically the utterances and gestures in the discourse that were related to these categories.

OBJECTIFYING THE RELATIONSHIP BETWEEN A FUNCTION AND ITS DERIVATIVE

The attention of the relationship between the lower graphs as a function and the upper graph as derivative

12 Mohamed: [while they drag f graph] Constant function parallel to x-axis [initially (Fig. 2) appears; then, as f is dragged vertically, there appears on the screen a constant function in the upper Cartesian system and a linear function in the lower Cartesian system (Fig. 3)]

13 Ahmed: This is a function [the lower graph] and that is its derivative [the upper graph].

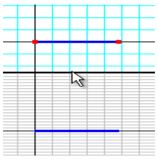


Fig 2

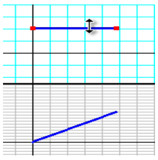


Fig 3

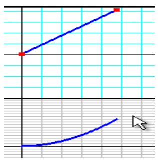
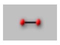


Fig 4

Fig 5

Fig 6

The students create a semiotic system, which contains two constant zero functions by the first icon  of the CIS [12]. When they drag f in the upper Cartesian system vertically they create a new semiotic system which contains a constant function in the upper Cartesian system and a non-constant linear function in the lower Cartesian system [12]. Immediately, the students declared a correlation between the two graphs – function and derivative. This suggests that the semiotic systems they create help them to attend to the correlation between the graphs, while their dragging of 'f' vertically and the corresponding change of 'g' suggest that the students become aware of the correlation between the slope of the linear function and the y-value of the constant function. As shown in the next transcript.

The awareness of the relationship between 'g' graph and 'f' graph

18 Ahmed: The slope is increasing [tracing the graph in the lower Cartesian system (Fig. 4) with the mouse] increasing, increasing

19 Mohamed: You must consider from here to here [trace the interval (0,1) on the x-axis in the lower Cartesian system]. How much is the slope? From here to here [indicates with the mouse on the x-axis from the origin up

to one, then goes up vertically to reach the graph in the lower system] it is about one.

20 Ahmed: It is about one [pointing to point its y-value is one on the upper Cartesian system] … how much here?

21 Mohamed: From here to here is two three… Ahh… It is increasing… the slope will behave like this [gestures with his hand (Fig. 5)] its slope is increasing [makes a gesture with two fingers, emulating the linear function (Fig. 6)]

In this excerpt, the students create a linear function in the upper Cartesian system and a quadratic function graph appears at the bottom. Ahmed initiates making sense of the connection between the two graphs. He describes the tangent slope of the quadratic by tracing it on the graph with the mouse and by using rhythm speech "increasing, increasing, increasing" [18]. The words [How much is the slope? From here to here] and the gestures [indicates with the mouse on the x-axis from the origin up to one, then goes up vertically to reach the graph] performed by Mohamed in [19] suggest that he is describing the rate of change of the quadratic function at several points. In [20] Ahmed initiates describing the connection between the two graphs for a specific value of x that the two graphs share. They correlate the tangent slope value in a point where its x coordinate is one in the lower graph to the y- value of the function in the upper graph for the same x coordinate [20]. Mohamed's gestures and words suggest that he is becoming aware of the behaviour of the tangent's slope of the quadratic function. Pointing to a point its y-value is one on the graph at the upper Cartesian, together with the gestures made by Mohamed – the first gesture signifying the tangent slope of the quadratic function and the second gesture signifying the linear function graph in the upper Cartesian system – which leads me to suggest that the students are becoming aware that the behaviour of tangent slope value of the lower function on points is the same behaviour of the y- value of the upper graph at the same x-coordinate [21].

OBJECTIFYING THE VERTICAL TRANSFORMATION OF 'G' GRAPH

The attention and awareness to dragging 'g' graph vertically

36 Mohamed: There is no difference between the slope of the graphs when we drag them up and down. We get the same slope if we drag it like that [drags 'g' vertically – (Fig. 7)]

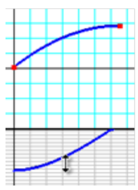


Fig 7

37 Ahmed: What is the value of this point? [The intersection point with the y-axes]

38 Mohamed: It is not the matter of the value of the point…The matter is considering the slope of the lower function, for example the slope is one [makes a motion with the mouse like a tangent line at a point on the lower graph] here is one [points toward the coordinate point in the upper graph]. The slope here is two [makes a motion with the mouse

like a tangent line at a point on the lower graph and simultaneously he points toward a coordinate point in the upper graph]

39	Ahmed:	I don't agree with you
40	Mohamed:	It is correct. The slope of the lower graph isn't changing even if you drag it [dragging 'g' vertically], the slope will not change

The students drag 'g' vertically. They attend that 'f' is not moving. Mohamed attends that the shape of 'g' does not change when they drag it vertically. Mohamed demonstrates his awareness of this fact by using the word 'slope'. His vertical dragging action, the words he is using, the gestures he is performing on the lower graph and the pointing gestures he is performing on the upper graph in [36] and [38] suggest that he is 'seeing' the tangent's slope of 'g' in different points and coordinates the it value to y-value of points on the upper system and he is repeating this process for other graphs which differ in a constant.

OBJECTIFYING THE RELATIONSHIP BETWEEN 'F' AND 'G'

First phase: Objectifying the function change

In this excerpt the students are discussing ways to describe how the tangent slope is behaving. While Ahmed claims that the slope of the tangent to the left of the minimum point is decreasing (Fig. 8) Mohamed is explaining how he sees the increase of the tangent slope.

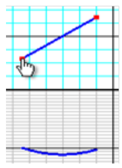


Fig 8

62	Mohamed:	It is minus two; minus one [y- values of 'f' from left to right] it is increasing [points to 'f'].
63	Ahmed:	It is the derivative [points to 'g']
64	Mohamed:	This graph ['f'] describes the slope of the function ['g']
65	Ahmed:	Wait a moment… here it is a function ['g']… how much is the slope? [Tracing the left side to the minimum point on 'g'] It is decreasing until it becomes zero
66	Mohamed:	It means increasing
67	Ahmed:	Correct! It is increasing [tracing 'f' until the intersection point of 'f' with x-axis]...when it becomes zero ['f'] it gets this point [he connects, with the mouse, the intersection point in the upper graph with the minimum point in 'g']
68	Mohamed:	When the slope is zero [traces with the mouse the minimum point of 'g'] it intersects x-axis [indicates the intersection point with x-axis in 'f']

The semiotic system of graphs, the gestures and the words in [62] suggest that Mohamed is considering, for the first time, the graph of 'f' and reading two y-coordinates as following each other, as a way to explain the increase of the tangent slope. While, Ahmed still maintains the view of derivative – function – the graph in the lower system is a function and in the upper system it is a derivative- mentioned and analysed in the last excerpt. Mohamed's declaration in [64] suggests that he has seen the connection between the y-values of the graph in the upper Cartesian system and the tangent slope value of the lower graph. In [65] Ahmed is checking Mohamed's claims by tracing the lower graph up to the minimum point. His tracing gesture imitates the behaviour of the tangent of the graph. The mismatch between Ahmed's tracing gesture and his statement "It is decreasing until it becomes zero" suggests that Ahmed has a difficulty describing the process of increasing a negative tangent slope [65, 66]. In [67] Ahmed leaves 'g' graph and concentrates on describing the upper graph, noticing that the intersection point between the upper graph and the x-axis corresponds to the minimum point in 'g' graph. In line [68] Mohamed explains that correspondence by identifying the slope of the minimum point as zero. This explanation suggests the role the word "slope" plays in correlating both graphs and in conceptualizing the idea of the antiderivative.

Second phase- objectifying the rate of change of 'f'

The students create by the fourth icon  an increasing function graph with a decreasing rate of change (concave down Fig. 10). In 'g', they obtained an increasing graph with an increasing rate of change (concave up). The students are confused about having two increasing graphs where one has an increasing rate of change ['g'] and the other, a decreasing rate of change ['f']. In the next excerpt, I analyse the processes and the emergence of the new semiotic means that enable them to solve this matter.

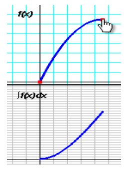


Fig 10

101 Ahmed: [Fig. 11 appears on the screen] It is the derivative [The graph in the upper system]

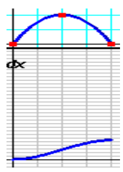


Fig 11

102 Mohamed: It is zero. Okay? Here it is two… If the value is two what does it means in the integral? The slope of the integral function at this point is two… What have we got here? Three. It is decreasing at three.

103 Ahmed: It is decreasing. The slope is decreasing

104 Mohamed: Look here [he deletes the right branch of Fig 11 and drags the right red point diagonally to get Fig. 10] [Silent for 10 seconds] I got the idea…wait a moment… when x is one the value is three [indicates the point (1,3) in the upper graph] Therefore, the slope is also three when x is one. [Points to lower graph] here it is four [points to the point (2,4) on the upper graph] thus the slope here is also four [makes a gesture like a segment on the lower graph]. Here its value is five plus [points to the point where x is 3 in the upper graph] thus its slope is five plus.

After they encountered difficulties interpreting the concavity of the integral graph [Fig. 10], the students made a new graph, which contained two different icons in the upper graph [Fig. 11]. The left icon with increasing change and decreasing rate of change, and the right icon, with decreasing change and increasing rate of change [Fig. 11]. Initially Ahmed signified the function graph as derivative [101]. Mohamed adapts Ahmed's claim as he is determining three y-values of the function graph. The students are well aware that each y-value in the function graph is the value of the slope of the tangent in the integral graph [102], and they are using the y-value of the function and the tangent slope of the integral function graph as means of semiotic mediation to explain the mathematical relationship between the two graphs. The use of these means is well illustrated in [104], where Mohamed interprets the relationship between the function graph and the integral graph.

DISCUSSION

Through the semiotic lens I determine the three essential foci of objectifying the antiderivative concept. The first focus is objectifying the relationship between a function and its derivative. The constant function icon plays a central role in objectifying the relationship between the graph in the lower Cartesian system and the graph in the upper Cartesian system. The vertical dragging action of the constant function graph performed by the students allows them to create an additional similar semiotic system, while linking the graphs according to the cultural meaning of the antiderivative helps them notice the difference in the similar semiotic systems. Once they notice that, the connection between the antiderivative graph slope value and the y-value of the constant function becomes apparent. Changing the semiotic system to include a linear function and quadratic antiderivative produced changes in the semiotic resources the students used to objectify the relationship between the function and the anti-derivative graph. Utilizing additional semiotic resources allows us to suggest that the slope of the graph, in the new situation, is not obvious to the students, unlike the constant function case. In order to make it obvious they use the rhythm speech to describe the tangent's slope behaviour, and the gesture to objectify the change of the tangent slope. The second focus considers the objectification of the vertical transformation of the antiderivative function. Dragging the antiderivative function graph vertically by the artifact and noticing that the function graph is fixed help the students become aware of an essential mathematical element of the antiderivative concept. The little variety of the semiotic resources used in this focus allow me to claim that objectifying the mathematical fact that the antiderivative of a function is a family of antiderivative functions is actually quite simple. The third focus considers the objectification of the antiderivative idea. That is, the y-value of a point on a function's graph is the tangent's slope value of the antiderivative function's graph for the same x-value. The current focus consists of two phases: objectifying the change of the function, objectifying the rate of change of the function. The first phase emerges through the students' confusion in describing the behaviour of a tangent slope of a decreasing graph with an increasing rate of change.

This confusion has helped them leave the bottom Cartesian system in favour of focusing on the top one. I assume that this happened because of the linearity of the function graph in the upper Cartesian system. The second phase does not refer to the shape of the anti-derivative function behaviour only, but does refer to how this form behaves. This phase emerged when the students created two contrasting graphs, one concave up and the other concave down.

REFERENCES

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. A. Jones, R. A. Lesh & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (Vol. second revised edition, pp. 746-783). Mahwah: Lawrence Erlbaum.

Berger, M. (2004). The functional use of a mathematical sign. *Educational Studies in Mathematics, 55*(1), 81-102. doi: 10.1023/B:EDUC.0000017672.49486.f2

Berry, J. S., & Nyman, M. A. (2003). Promoting students' graphical understanding of the calculus. *The Journal of Mathematical Behavior, 22*(4), 479-495. doi: 10.1016/j.jmathb.2003.09.006

Haciomeroglu, E. S., Aspinwall, L., & Presmeg, N. C. (2010). Contrasting Cases of Calculus Students' Understanding of Derivative Graphs. *Mathematical Thinking and Learning, 12*(2), 152-176. doi: 10.1080/10986060903480300

Radford, L. (2003). Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization. *Mathematical Thinking and Learning, 5*(1), 37-70. doi: 10.1207/s15327833mtl0501_02

Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the general the multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education, 38*(5), 507-530.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th conference of the international group for the Psychology of Mathematics Education* (Vol. 4, pp. 113-120). Australia: University of Melbourne.

Shternberg, B., Yerushalmy, M., & Zilber, A. (Producer). (2004). The Calculus Integral Sketcher. Retrieved from http://calculus.cet.ac.il/Lib/index.aspx (substitution by http://calculus.cet.ac.il/Lib/item.aspx?sID=DDE91E48-EB26-41B6-ABE3-BACBADBCEA78&bPopup=1&bFitSize=1)

Thompson, P. W., Byerley, C., & Hatfield, N. (in press). A conceptual approach to calculus made possible by technology *Computers in the Schools*.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America.

Yerushalmy, M., & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representation environment. *Educational Studies in Mathematics*, 1-20. doi: 10.1007/s10649-011-9356-8